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Field-dependent BRST–antiBRST transformations in Yang–Mills and Gribov–Zwanziger theories

Pavel Yu. Moshin [a], Alexander A. Reshetnyak [b,c,*]

[a] *Department of Physics, Tomsk State University, 634050, Russia*

[b] *Institute of Strength Physics and Materials Science, Siberian Branch of Russian Academy of Sciences, 634021, Tomsk, Russia*

[c] *Tomsk State Pedagogical University, 634061, Russia*

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Abstract

We introduce the notion of finite BRST–antiBRST transformations, both global and field-dependent, with a doublet λ_a, $a = 1, 2$, of anticommuting Grassmann parameters and find explicit Jacobians corresponding to these changes of variables in Yang–Mills theories. It turns out that the finite transformations are quadratic in their parameters. At the same time, exactly as in the case of finite field-dependent BRST transformations for the Yang–Mills vacuum functional, special field-dependent BRST–antiBRST transformations, with s_a-potential parameters $\lambda_a = s_a \Lambda$ induced by a finite even-valued functional Λ and by the anticommuting generators s_a of BRST–antiBRST transformations, amount to a precise change of the gauge-fixing functional. This proves the independence of the vacuum functional under such BRST–antiBRST transformations. We present the form of transformation parameters that generates a change of the gauge in the path integral and evaluate it explicitly for connecting two arbitrary R_ξ-like gauges. For arbitrary differentiable gauges, the finite field-dependent BRST–antiBRST transformations are used to generalize the Gribov horizon functional h, given in the Landau gauge, and being an additive extension of the Yang–Mills action by the Gribov horizon functional in the Gribov–Zwanziger model. This generalization is achieved in a manner consistent with the study of gauge independence. We also discuss an extension of finite BRST–antiBRST transformations to the case of general gauge theories and present an ansatz for such transformations.

* Corresponding author.
E-mail addresses: moshin@rambler.ru (P.Yu. Moshin), reshet@ispms.tsc.ru (A.A. Reshetnyak).

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1. Introduction

Contemporary quantization methods for gauge theories [1–4] are based primarily on the special supersymmetries known as BRST symmetry [5–7] and BRST–antiBRST symmetry [8–11]. They are characterized by the presence of a Grassmann-odd parameter μ and two Grassmann-odd parameters $(\mu, \bar{\mu})$, respectively. In the framework of the Sp(2)-covariant schemes of generalized Hamiltonian [12,13] and Lagrangian [15,16] quantization (see also [14,18]), the parameters $(\mu, \bar{\mu}) \equiv (\mu_1, \mu_2) = \mu_a$ form an Sp(2)-doublet. These infinitesimal odd-valued parameters may be regarded as constants and thus used to derive the Ward identities. They may also be chosen as field-dependent functionals and thus used to establish the gauge-independence of the corresponding vacuum functional in the path integral approach.

BRST transformations with a finite field-dependent parameter in Yang–Mills theories, whose quantum action is constructed by the Faddeev–Popov rules [19], were first introduced in [20] by means of a functional equation for the parameter in question, and used to provide the path integral with such a change of variables that would allow one to relate the quantum action in a certain gauge with the one given in a different gauge; see also [21]. This equation, as well as a similar equation [22] for the finite parameter of a field-dependent BRST transformation in the generalized Hamiltonian formalism, has not been solved in the general setting of the problem. Namely, the corresponding equation (4.13) in [20], or Eq. (3.6) in [22], for the Jacobian J of a change of variables given by infinitesimal field-dependent BRST transformations with an odd-valued functional[1] $\Theta'(\phi(\kappa))$ allows one to express an additional contribution S_1 to the quantum action in terms of $\Theta(\phi(0))$, but has not been solved neither in the form $S_1 = S_1(\Theta(\phi(0)))$, for an unknown quantity S_1, nor in the form $S_1 = S_1(\Theta(\phi(0)))$, for an unknown quantity $\Theta(\phi(0))$. Instead, a series of particular cases having the form of an ansatz for the functional S_1 have been examined, and a solution of the above-mentioned equation was found without any explicit calculation of the Jacobian for the change of variables induced by finite field-dependent BRST transformations.[2] On the other hand, there emerges the problem of establishing a relation of the Faddeev–Popov action in a certain gauge with the action in a different gauge, by using a change of variables induced by a finite field-dependent BRST transformation. This problem was solved for the first time in the case of linear and quadratic gauges in [20] and for the class of general gauges in [23], thereby providing an exact relation between a finite parameter and a finite variation of the gauge-fixing condition in terms of the gauge Fermion. There it was established that the Jacobian of any finite field-dependent BRST transformation reproduces BRST-exact terms, which can be entirely absorbed into the gauge-fixed part the of BRST-invariant Faddeev–Popov action, corresponding to a certain change of the gauge $\Delta\psi$, so that the vacuum functional $Z_{\psi+\Delta\psi}$, resulting from the above change of variables, coincides with the initial vacuum functional Z_ψ and should be regarded as a vacuum functional with the same BRST-exact classical (renormalized) action,

[1] $\Theta'(\phi(\kappa))$ depends on a numerical parameter, κ, so that the finite field-dependent BRST transformations with the odd-valued functional $\Theta(\phi(0))$ are obtained from $\Theta'(\phi(\kappa))$ by $\Theta(\phi(0)) = \int_0^1 \Theta'(\phi(\kappa))dk$.

[2] The property of gauge independence for the vacuum functional in the Yang–Mills theory with an action constructed by the Faddeev–Popov recipe [19], or with an action constructed by the Batalin–Vilkovisky (BV) procedure [30], uses an explicit form of the above Jacobian.

having, however, a gauge-fixed (BRST-exact) action given by a different gauge, $\psi + \Delta\psi$. In particular, this implies the conservation of the number of physical degrees of freedom in a given Yang–Mills theory with respect to finite field-dependent BRST transformations. This means the impossibility of relating the Yang–Mills theory to a theory whose action may contain, in addition to the Faddeev–Popov action, some BRST non-invariant terms (such as the Gribov horizon functional in the Gribov–Zwanziger theory [34], having additional degrees of freedom as compared to the Yang–Mills theory) in the same configuration space.[3]

The solution of a similar problem for arbitrary dynamical systems with first-class constraints in the generalized Hamiltonian formalism [7,27,28] has been recently proposed in [29]. For general gauge theories, which may possess a reducible gauge symmetry and/or an open gauge algebra, an exact Jacobian corresponding to a change of variables given by field-dependent BRST transformations in the path integral constructed according to the Batalin–Vilkovisky (BV) procedure [30] was obtained in [31] and shown to be identical with the Jacobian of the Yang–Mills theory. The study of [31] extends the results of [23] to first-rank theories with a closed algebra and solves the problem of gauge-independence for gauge theories with the so-called soft breaking of BRST symmetry. This problem was raised in [32] to study the problem of Gribov copies [33] by using various gauges in the Gribov–Zwanziger approach [34]; for recent progress, see [35–39].

On the other hand, there emerges the problem of finding a correspondence of the quantum action in the BRST–antiBRST invariant Lagrangian quantization [15–17], where gauge is introduced by a Bosonic gauge-fixing functional, F, with the quantum action of the same theory in a different gauge, $F + \Delta F$, for a finite value ΔF, by using a change of variables in the vacuum functional. This problem has not been solved even in theories of Yang–Mills type. Note that finite field-dependent antiBRST transformations in Yang–Mills theories were considered in [25] in the same way as in the case of BRST transformations [20], so as to relate the antiBRST invariant quantum action of a Yang–Mills theory in different gauges by using an ansatz for a term introduced to the quantum action in order to satisfy an infinitesimal functional equation for the transformation parameter. The study of [26] proposed finite two-parametric BRST–antiBRST transformations ("mixed", by the terminology [26]): "$\delta_m \phi = \overleftarrow{s}_{\mathrm{a}} \Theta_1 + \overleftarrow{s}_{\mathrm{ab}} \Theta_2$" in (3.7), including field-dependent ones, which form a Lie superalgebra; however, without any parameters, constant and/or field-dependent, being quadratic in Θ_1, Θ_2 (allowing one to consider BRST–antiBRST transformations as group transformations), which prohibits the complete BRST–antiBRST invariance of the quantum action in Yang–Mills theories and similarly in more general gauge theories. Therefore, this leads immediately to the problem of finding a solution for the above functional equation, since the latter does not "feel" the finite polynomial character of the parameters $\Theta_1 \cdot \Theta_2$, and therefore prohibits the gauge independence of the vacuum functional under finite field-dependent BRST–antiBRST transformations even for functionally-dependent parameters (see footnote 6).

A similar problem in the Sp(2)-covariant generalized Hamiltonian formalism [12,13] remains unsolved[4] as well. We expect that the solution of these problems in the Lagrangian and Hamiltonian quantization schemes for gauge theories should be based on the concept of finite BRST–antiBRST transformations with an Sp(2)-doublet of Grassmann-odd parameters $\mu_a(\phi)$

[3] Instead of a local Gribov–Zwanziger horizon functional S_γ in (3.3), there exists a relation [24] by finite field-dependent BRST transformations to a BRST-invariant model with the functional Σ_γ in (3.6), being a Yang–Mills theory defined in an appropriate configuration space.

[4] For the recent progress achieved in this area since the appearance of the present work in arXiv, see footnote 11 in Discussion.

depending on the field variables. This would allow one to generate the Gribov horizon functional by using different gauges in a way consistent with the gauge-independence of the path integral, based on the Gribov–Zwanziger prescription [34] and starting from the BRST–antiBRST invariant Yang–Mills quantum action in the Landau gauge.

Motivated by these reasons, we intend to address the following issues, paying our attention primarily to the Yang–Mills theory in Lagrangian formalism:

1. introduction of *finite BRST–antiBRST transformations*, being polynomial in powers of a constant Sp(2)-doublet of Grassmann-odd parameters λ_a and leaving the quantum action of the Yang–Mills theory invariant to all orders in λ_a;
2. definition of *finite field-dependent BRST–antiBRST transformations*, being polynomial in powers of an Sp(2)-doublet of Grassmann-odd functionals $\lambda_a(\phi)$ depending on the classical Yang–Mills fields, the ghost–antighost fields, and the Nakanishi–Lautrup fields; calculation of the Jacobian related to a change of variables by using a special class of such transformations with s_a-potential parameters $\lambda_a(\phi) = s_a \Lambda(\phi)$ for a Grassmann-even functional $\Lambda(\phi)$ and Grassmann-odd generators s_a of BRST–antiBRST transformations;
3. solution of the so-called compensation equation for an unknown functional Λ generating the Sp(2)-doublet λ_a with the purpose of establishing a relation of the Yang–Mills quantum action S_F in a gauge determined by a gauge Boson F with the quantum action $S_{F+\Delta F}$ in a different gauge $F + \Delta F$;
4. explicit construction of the parameters λ_a of finite field-dependent BRST–antiBRST transformations generating a change of the gauge in the path integral within a class of linear R_ξ-like gauges realized in terms of Bosonic gauge functionals $F_{(\xi)}$, with $\xi = 0, 1$ corresponding to the Landau and Feynman (covariant) gauges, respectively;
5. construction of the Gribov horizon functional h_ξ in arbitrary R_ξ-like gauges by means of finite field-dependent BRST–antiBRST transformations starting from a known BRST–antiBRST non-invariant functional h, given in the Landau gauge and realized in terms of the Bosonic functional $F_{(0)}$.

The present work is organized as follows. In Section 2, we remind the general setup of the BRST–antiBRST Lagrangian quantization of general gauge theories and list its basics ingredients. In Section 3, we introduce the notion of finite BRST–antiBRST transformations, both global and local (field-dependent). We find an explicit Jacobian corresponding to this change of variables in theories of Yang–Mills type and show that, exactly as in the case of field-dependent BRST transformations for the Yang–Mills vacuum functional [23], the field-dependent transformations amount to a precise change of the gauge-fixing functional. In Section 4, we present the form of transformation parameters that generates a change of the gauge and evaluate it for connecting two arbitrary R_ξ-like gauges in Yang–Mills theories. In Section 5, the Gribov horizon functional in an arbitrary R_ξ-like gauge, and generally in any differentiable gauge, is determined with the help of respective finite field-dependent BRST–antiBRST transformations. In Discussion, we make an overview of our results and outline some open problems. In particular, we discuss an extension of finite BRST–antiBRST transformations to the case of general gauge theories and present an ansatz for such transformations. In Appendix A, we study the group properties of finite field-dependent BRST–antiBRST transformations. In Appendix B, we present a detailed calculation of the Jacobian corresponding to the finite, both global (Appendix B.1) and field-dependent (Appendix B.2), BRST–antiBRST transformations. Appendix C is devoted to calculations involving the BRST–antiBRST invariant Yang–Mills action in R_ξ-gauges.

We use DeWitt's condensed notations [40]. By default, derivatives with respect to the fields are taken from the right, and those with respect to the corresponding antifields are taken from the left; otherwise, left-hand and right-hand derivatives are labeled by the subscripts "l" and "r", respectively; $F_{,A}$ stands for the right-hand derivative $\delta F/\delta\phi^A$ of a functional $F = F(\phi)$ with respect to ϕ^A. The raising and lowering of Sp(2) indices, $s^a = \varepsilon^{ab}s_b$, $s_a = \varepsilon_{ab}s^b$, is carried out with the help of a constant antisymmetric second-rank tensor ε^{ab}, $\varepsilon^{ac}\varepsilon_{cb} = \delta^a_b$, subject to the normalization condition $\varepsilon^{12} = 1$. The Grassmann parity and ghost number of a quantity A, assumed to be homogeneous with respect to these characteristics, are denoted by $\varepsilon(A)$, $\mathrm{gh}(A)$, respectively.

2. General setup for BRST–antiBRST Lagrangian quantization

The BRST–antiBRST Lagrangian quantization of general gauge theories [15–17] involves a set of fields ϕ^A and a set of corresponding antifields ϕ^*_{Aa} $(a = 1, 2)$, $\bar{\phi}_A$, where the doublets of antifields ϕ^*_{Aa} play the role of sources to the BRST and antiBRST transformations, while the antifields $\bar{\phi}_A$ are the sources to the mixed BRST and antiBRST transformations, with the following distributions of the Grassmann parity and ghost number:

$$\begin{aligned} &\varepsilon\big(\phi^A\big) \equiv \varepsilon_A, \qquad \varepsilon\big(\phi^*_{Aa}\big) = \varepsilon_A + 1, \qquad \varepsilon(\bar{\phi}_A) = \varepsilon_A, \\ &\mathrm{gh}\big(\phi^*_{Aa}\big) = (-1)^a - \mathrm{gh}\big(\phi^A\big), \qquad \mathrm{gh}(\bar{\phi}_A) = -\mathrm{gh}\big(\phi^A\big). \end{aligned} \tag{2.1}$$

The configuration space of fields ϕ^A is identical with that of the BV formalism [30] of covariant quantization and is determined by the properties of the initial classical theory. Namely, we consider an initial classical theory of fields A^i, $\varepsilon(A^i) \equiv \varepsilon_i$, with an action $S_0(A)$ invariant under gauge transformations,

$$\delta A^i = R^i_{\alpha_0}(A)\zeta^{\alpha_0} \quad \Longrightarrow \quad S_{0,i}(A)R^i_{\alpha_0}(A) = 0, \tag{2.2}$$

where $R^i_{\alpha_0}(A)$ are generators of the gauge transformations, $\varepsilon(R^i_{\alpha_0}) = \varepsilon_i + \varepsilon_{\alpha_0}$, and ζ^{α_0} are arbitrary functions of the space–time coordinates, $\varepsilon(\zeta^{\alpha_0}) = \varepsilon_{\alpha_0}$. The generators $R^i_{\alpha_0}(A)$ form a gauge algebra [30] with the relations

$$\begin{aligned} &R^i_{\alpha_0,j}(A)R^j_{\beta_0}(A) - (-1)^{\varepsilon_{\alpha_0}\varepsilon_{\beta_0}}R^i_{\beta_0,j}(A)R^j_{\alpha_0}(A) \\ &\quad = -R^i_{\gamma_0}(A)F^{\gamma_0}_{\alpha_0\beta_0}(A) - S_{0,j}(A)M^{ij}_{\alpha_0\beta_0}(A), \\ &F^{\gamma_0}_{\alpha_0\beta_0} = -(-1)^{\varepsilon_{\alpha_0}\varepsilon_{\beta_0}}F^{\gamma_0}_{\beta_0\alpha_0}, \qquad M^{ij}_{\alpha_0\beta_0} = -(-1)^{\varepsilon_i\varepsilon_j}M^{ji}_{\alpha_0\beta_0} = -(-1)^{\varepsilon_{\alpha_0}\varepsilon_{\beta_0}}M^{ij}_{\beta_0\alpha_0}. \end{aligned} \tag{2.3}$$

In case the vectors $R^i_{\alpha_0}(A)$, enumerated by the index α_0, are linearly independent, the theory is irreducible; otherwise it is reducible. Depending on the (ir)reducibility of the generators of gauge transformations, the specific structure of the configuration space ϕ^A is described by the set of fields

$$\phi^A = \big(A^i, B^{\alpha_s|a_1\ldots a_s}, C^{\alpha_s|a_0\ldots a_s}\big), \quad s = 0, 1, \ldots, L, \tag{2.4}$$

where the ghost $C^{\alpha_s|a_0\ldots a_s}$ and auxiliary $B^{\alpha_s|a_1\ldots a_s}$ fields form symmetric Sp(2) tensors, being irreducible representations of the Sp(2) group, with the corresponding distribution [16] of the Grassmann parity and ghost number. These fields absorb the pyramids of ghost–antighost and Nakanishi–Lautrup fields of a given (ir)reducible gauge theory, where L in (2.4) is the corresponding stage of reducibility [30], and $L = 0$ stands for an irreducible theory.

In the space of fields and antifields $(\phi^A, \phi^*_{Aa}, \bar{\phi}_A)$, one introduces the basic object of the BRST–antiBRST Lagrangian scheme, being an even-valued functional $S = S(\phi, \phi^*, \bar{\phi})$ subject to an Sp(2)-doublet of the generating equations [15]

$$\frac{1}{2}(S, S)^a + V^a S = i\hbar\Delta^a S \quad \Longleftrightarrow \quad \bar{\Delta}^a \exp\big[(i/\hbar)S\big] = 0,$$
$$\bar{\Delta}^a = \Delta^a + (i/\hbar)V^a. \tag{2.5}$$

Here, $\hbar$ is the Planck constant, whereas the extended antibracket $(\cdot,\cdot)^a$ and the operators Δ^a, V^a are given by

$$(\cdot,\cdot)^a = \frac{\delta_r\cdot}{\delta\phi^A}\frac{\delta_l\cdot}{\delta\phi^*_{Aa}} - \frac{\delta_r\cdot}{\delta\phi^*_{Aa}}\frac{\delta_l\cdot}{\delta\phi^A}, \qquad \Delta^a = (-1)^{\varepsilon_A}\frac{\delta_l}{\delta\phi^A}\frac{\delta}{\delta\phi^*_{Aa}},$$
$$V^a = \varepsilon^{ab}\phi^*_{Ab}\frac{\delta}{\delta\bar{\phi}_A}. \tag{2.6}$$

The properties of the operators Δ^a, V^a, $\bar{\Delta}^a$ and those of the extended antibracket $(\cdot,\cdot)^a$ were investigated in [15]. The study of [17] proved the existence of solutions to (2.5) with the boundary condition $S|_{\phi^*=\bar{\phi}=\hbar=0} = S_0$ in the form of an expansion in powers of $\hbar$ and described the arbitrariness in solutions, which is controlled by a transformation generated by the operators $\bar{\Delta}^a$, connecting two solutions and describing the gauge-fixing procedure. A solution $S = S(\phi, \phi^*, \bar{\phi})$ of the generating equations (2.5) allows one to construct an extended (due to the antifields) generating functional of Green's functions $Z(J, \phi^*, \bar{\phi})$ for the fields ϕ^A of the total configuration space [15], namely,

$$Z\big(J, \phi^*, \bar{\phi}\big) = \int d\phi \exp\left\{\frac{i}{\hbar}\big[S_{\rm ext}\big(\phi, \phi^*, \bar{\phi}\big) + J_A\phi^A\big]\right\}. \tag{2.7}$$

Hence, the generating functional of Green's functions $Z(J) = Z(J, \phi^*, \bar{\phi})|_{\phi^*=\bar{\phi}=0}$ is given by

$$Z(J) = \int d\phi \exp\left\{\frac{i}{\hbar}\big[S_{\rm eff}(\phi) + J_A\phi^A\big]\right\}, \quad \text{with } S_{\rm eff}(\phi) = S_{\rm ext}\big(\phi, \phi^*, \bar{\phi}\big)\big|_{\phi^*=\bar{\phi}=0}, \tag{2.8}$$

where J_A, $\varepsilon(J_A) = \varepsilon_A$, are external sources to the fields ϕ^A, and $S_{\rm ext} = S_{\rm ext}(\phi, \phi^*, \bar{\phi})$ is an action constructed with the help of an even-valued gauge-fixing functional $F = F(\phi)$:

$$\exp\big[(i/\hbar)S_{\rm ext}\big] = \hat{U}\exp\big[(i/\hbar)S\big],$$
$$\text{with } \hat{U} = \exp\left(F_{,A}\frac{\delta}{\delta\bar{\phi}_A} + \frac{i\hbar}{2}\varepsilon_{ab}\frac{\delta}{\delta\phi^*_{Aa}}F_{,AB}\frac{\delta}{\delta\phi^*_{Bb}}\right). \tag{2.9}$$

Due to the commutativity of $\bar{\Delta}^a$ and $\hat{U}$, the gauge-fixing procedure retains the form of the generating equations (2.5),

$$\bar{\Delta}^a \exp\big[(i/\hbar)S_{\rm ext}\big] = 0. \tag{2.10}$$

A possible choice of the gauge-fixing functional $F(\phi)$ has the form of the most general Sp(2)-scalar being quadratic in the ghost and auxiliary fields [16].

Introducing a set of auxiliary fields π^{Aa} and λ^A,

$$\varepsilon\big(\pi^{Aa}\big) = \varepsilon_A + 1, \qquad \varepsilon\big(\lambda^A\big) = \varepsilon_A,$$
$$\mathrm{gh}\big(\pi^{Aa}\big) = -(-1)^a + \mathrm{gh}\big(\phi^A\big), \qquad \mathrm{gh}\big(\lambda^A\big) = \mathrm{gh}\big(\phi^A\big), \tag{2.11}$$

one can represent $Z(J)$ as a functional integral in the extended space of variables [15]

$$Z(J) = \int d\Gamma \exp\left\{\frac{i}{\hbar}\left[S + \phi^*_{Aa}\pi^{Aa} + (\bar{\phi}_A - F_{,A})\lambda^A \right.\right.$$
$$\left.\left. - (1/2)\varepsilon_{ab}\pi^{Aa}F_{,AB}\pi^{Bb} + J_A\phi^A\right]\right\}, \tag{2.12}$$

where $d\Gamma = d\phi\, d\phi^*\, d\bar{\phi}\, d\lambda\, d\pi$ is the integration measure.

An important property of the integrand in (2.12) for $J_A = 0$ is its invariance under the following infinitesimal transformations of global supersymmetry:

$$\delta\left(\phi^A, \phi^*_{Aa}, \bar{\phi}_A, \pi^{Aa}, \lambda^A\right) = \left(\pi^{Aa}\mu_a, \mu_a S_{,A}, \varepsilon^{ab}\mu_a\phi^*_{Ab}, -\varepsilon^{ab}\lambda^A\mu_b, 0\right), \tag{2.13}$$

where μ_a is a doublet of constant anticommuting Grassmann parameters, $\mu_a\mu_b + \mu_b\mu_a \equiv 0$. The transformations (2.13) realize the BRST–antiBRST transformations in the extended space $(\phi^A, \phi^*_{Aa}, \bar{\phi}_A, \pi^{Aa}, \lambda^A)$.

The symmetry of the integrand in (2.12) for $J_A = 0$ under the transformations (2.13) with constant infinitesimal μ_a allows one to derive the following Ward identities in the extended space:

$$J_A\left\langle\pi^{Aa}\right\rangle_{F,J} = 0,$$
$$\text{for } \langle\mathcal{O}\rangle_{F,J} = Z^{-1}(J)\int d\Gamma\, \mathcal{O} \exp\left\{\frac{i}{\hbar}\left[S + \phi^*_{Aa}\pi^{Aa} + (\bar{\phi}_A - F_{,A})\lambda^A \right.\right.$$
$$\left.\left. - (1/2)\varepsilon_{ab}\pi^{Aa}F_{,AB}\pi^{Bb} + J_A\phi^A\right]\right\}, \tag{2.14}$$

where the expectation value of a functional $\mathcal{O}(\Gamma)$ is given in the extended space parameterized by Γ with a gauge $F(\phi)$ in the presence of external sources J_A. To obtain (2.14), we subject (2.12) to a change of variables $\Gamma \to \Gamma + \delta\Gamma$ with $\delta\Gamma$ given by (2.13) and use Eqs. (2.5) for S. At the same time, with allowance for the equivalence theorem [41], the transformations (2.13) permit one to establish the independence of the S-matrix from the choice of a gauge. Indeed, suppose $Z_F \equiv Z(0)$ and change the gauge, $F \to F + \Delta F$, by an infinitesimal value ΔF. In the functional integral for $Z_{F+\Delta F}$ we now make the change of variables (2.13). Then, choosing the parameters μ_a as

$$\mu_a = -\frac{i}{2\hbar}\varepsilon_{ab}(\Delta F)_{,A}\pi^{Ab}, \tag{2.15}$$

we find that $Z_{F+\Delta F} = Z_F$, and therefore the S-matrix is gauge-independent.

For the purpose of a subsequent treatment of Yang–Mills theories, we need the particular case of solutions to the generating equations (2.5) given by a functional $S = S(\phi, \phi^*, \bar{\phi})$ linear in the antifields. Namely, we assume

$$S = S_0 + \phi^*_{Aa}X^{Aa} + \bar{\phi}_A Y^A, \tag{2.16}$$

which implies

$$S_{0,i}X^{ia} = 0, \qquad X^{Aa}_{,B}X^{Bb} = \varepsilon^{ab}Y^A, \qquad Y^B_{,A}X^{Aa} = 0, \qquad X^{Aa}_{,A} = 0 \tag{2.17}$$

and allows one to present S in the form

$$S = S_0 + \phi^*_{Aa}\left(s^a\phi^A\right) - \frac{1}{2}\bar{\phi}_A\left(s^2\phi^A\right), \quad s^2 \equiv s_a s^a, \tag{2.18}$$

where s^a are generators of BRST–antiBRST transformations,

$$\delta\phi^A = \left(s^a\phi^A\right)\mu_a, \quad s^a\phi^A = X^{Aa}, \tag{2.19}$$

and s^2 are generators of mixed BRST–antiBRST transformations,

$$\delta^2\phi^A = s^a\left(s^b\phi^A\mu_b\right)\mu_a = -\frac{1}{2}\left(s^2\phi^A\right)\mu^2, \quad s^2\phi^A = \varepsilon_{ab}X^{Aa}_{,B}X^{Bb} = -2Y^A. \tag{2.20}$$

The explicit form of X^{Aa} and Y^A for theories of Yang–Mills type was found in [15] and is given in Appendix C.

For a solution of (2.5) linear in the antifields, integration in (2.12) over ϕ^*_{Aa}, $\bar{\phi}_A$, π^{Aa}, λ^A is trivial [15]:

$$Z(J) = \int d\phi \exp\left\{\frac{i}{\hbar}\left[S_F(\phi) + J_A\phi^A\right]\right\}, \tag{2.21}$$

where

$$S_F(\phi) = S_0(A) + F_{,A}Y^A - (1/2)\varepsilon_{ab}X^{Aa}F_{,AB}X^{Bb}, \tag{2.22}$$

which can also be established directly by inserting the solution (2.16) into (2.9).

The quantum action $S_F(\phi)$ can be presented in terms of a mixed BRST–antiBRST variation,

$$S_F(\phi) = S_0(A) - (1/2)s^2F(\phi), \tag{2.23}$$

where the operators s^a, acting on an arbitrary functional $V = V(\phi)$ of any Grassmann parity, define a BRST–antiBRST analogue of the Slavnov variation, $s^aV = V_{,A}(s^a\phi^A)$. Thus defined operators s^a are anticommuting, $s^as^b + s^bs^a \equiv 0$, for any $a, b = 1, 2$,

$$\begin{aligned} &s^as^bV = \varepsilon^{ab}W, \qquad W \equiv (1/2)\varepsilon_{ab}V_{,BA}X^{Aa}X^{Bb}(-1)^{\varepsilon_B} - V_{,A}Y^A, \\ &s^as^bV = (1/2)\varepsilon^{ab}s^2V, \qquad W = (1/2)s^2V, \end{aligned} \tag{2.24}$$

and therefore nilpotent, $s^as^bs^c \equiv 0$, which proves the invariance of S_F given by (2.23) under the infinitesimal transformations (2.19),

$$\delta S_F = (S_F)_{,A}\delta\phi^A = \left(s^aS_F\right)\mu_a = \left(s^aS_0\right)\mu_a - \frac{1}{2}\left(s^as^2F\right)\mu_a = 0,$$

by virtue of the condition $s^aS_0 = S_{0,i}X^{ia} = 0$ from (2.17), being a consequence of the Noether identities (2.2).

In view of the condition $X^{Aa}_{,A} = 0$ from (2.17), the integration measure in (2.21) is also invariant under the transformations (2.19), which ensures the invariance of the integrand in (2.21) for $J_A = 0$ under (2.19). By analogy with the previous consideration, this allows one to establish the Ward identities for $Z(J)$ in (2.21),

$$\begin{aligned} &J_A\left\langle s^a\phi^A\right\rangle_{F,J} = J_A\left\langle X^{Aa}(\phi)\right\rangle_{F,J} = 0 \\ &\quad \text{for } \langle\mathcal{O}\rangle_{F,J} = Z^{-1}(J)\int d\phi\, \mathcal{O}(\phi)\exp\left\{\frac{i}{\hbar}\left[S_F(\phi) + J_A\phi^A\right]\right\}, \end{aligned} \tag{2.25}$$

as well as the independence of the S-matrix from the choice of a gauge. Indeed, suppose $Z_F \equiv Z(0)$ in (2.21) and change the gauge $F \to F + \Delta F$ by an infinitesimal value ΔF. Then, making in $Z_{F+\Delta F}$ the change of variables (2.19) with the field-dependent infinitesimal parameters

$$\mu_a = \frac{i}{2\hbar}\varepsilon_{ab}(\Delta F)_{,A}X^{Ab} = \frac{i}{2\hbar}(s_a\Delta F), \tag{2.26}$$

being a particular case of the field-dependent BRST–antiBRST transformations studied in the following section, we find $Z_{F+\Delta F} = Z_F$, which establishes the gauge-independence of the S-matrix.

3. Finite BRST–antiBRST transformations and their Jacobians

Let us introduce finite transformations of the fields ϕ^A with a doublet λ_a of anticommuting Grassmann parameters, $\lambda_a\lambda_b + \lambda_b\lambda_a = 0$,

$$\phi^A \to \phi'^A = \phi^A + \Delta\phi^A = \phi'^A(\phi|\lambda), \quad \text{so that} \quad \phi'^A(\phi|0) = \phi^A. \tag{3.1}$$

In the general case, such transformations are quadratic in the parameters, due to $\lambda_a\lambda_b\lambda_c \equiv 0$,

$$\phi'^A(\phi|\lambda) = \phi'^A(\phi|0) + \left[\frac{\overleftarrow{\partial}}{\partial\lambda_a}\phi'^A(\phi|\lambda)\right]_{\lambda=0}\lambda_a + \frac{1}{2}\left[\frac{\overleftarrow{\partial}}{\partial\lambda_a}\frac{\overleftarrow{\partial}}{\partial\lambda_b}\phi'^A(\phi|\lambda)\right]\lambda_a\lambda_b, \tag{3.2}$$

which implies

$$\Delta\phi^A = Z^{Aa}\lambda_a + (1/2)Z^A\lambda^2, \quad \text{where } \lambda^2 \equiv \lambda_a\lambda^a, \tag{3.3}$$

for certain functions $Z^{Aa} = Z^{Aa}(\phi)$, $Z^A = Z^A(\phi)$, corresponding to the first- and second-order derivatives of $\phi'^A(\phi|\lambda)$ with respect to λ_a in (3.2).

In view of the obvious property of nilpotency $\Delta\phi^{A_1}\cdots\Delta\phi^{A_n} \equiv 0$, $n \geq 3$, an arbitrary functional $F(\phi)$ under the above transformations $\phi^A \to \phi^A + \Delta\phi^A$ can be expanded as

$$F(\phi + \Delta\phi) = F(\phi) + F_{,A}(\phi)\Delta\phi^A + (1/2)F_{,AB}(\phi)\Delta\phi^B\Delta\phi^A. \tag{3.4}$$

Based on (3.1)–(3.4), we now introduce *finite BRST–antiBRST transformations* as invariance transformations of the quantum action $S_F(\phi)$ given by (2.23) under finite transformations of the fields ϕ^A, such that

$$S_F(\phi + \Delta\phi) = S_F(\phi),$$
$$\left[\frac{\overleftarrow{\partial}}{\partial\lambda_a}\Delta\phi^A\right]_{\lambda=0} = s^a\phi^A \quad \text{and} \quad \left[\frac{\overleftarrow{\partial}}{\partial\lambda_a}\frac{\overleftarrow{\partial}}{\partial\lambda_b}\Delta\phi^A\right] = \frac{1}{2}\varepsilon^{ab}s^2\phi^A, \tag{3.5}$$

which implies $Z^{Aa} = s^a\phi^A = X^{Aa}$ and $Z^A = (1/2)s^2\phi^A = -Y^A$, according to (2.19), (2.20), (3.3).

One can easily verify the consistency of definition (3.5) by considering the equation, implied by $\Delta S_F = 0$,

$$(S_F)_{,A}\left(X^{Aa}\lambda_a - \frac{1}{2}Y^A\lambda^2\right) + \frac{1}{2}(S_F)_{,AB}\left(X^{Bb}\lambda_b - \frac{1}{2}Y^B\lambda^2\right)\left(X^{Aa}\lambda_a - \frac{1}{2}Y^A\lambda^2\right) = 0. \tag{3.6}$$

Taking into account the fact $\lambda_a\lambda^2 = \lambda^4 \equiv 0$, the invariance relations $(S_F)_{,A}X^{Aa} = 0$, and their differential consequences $(S_F)_{,AB}X^{Bb}\lambda_bX^{Aa}\lambda_a = (S_F)_{,A}Y^A\lambda^2$, implied by the relations $Y^A = (1/2)X^{Aa}_{,B}X^{Bb}\varepsilon_{ba}$ from (2.20), we find that the above equation is satisfied identically:

$$(S_F)_{,A}X^{Aa}\lambda_a - \frac{1}{2}(S_F)_{,A}Y^A\lambda^2 + \frac{1}{2}(S_F)_{,AB}X^{Bb}\lambda_bX^{Aa}\lambda_a \equiv 0.$$

Explicitly, the finite BRST–antiBRST transformations can be presented as[5]

$$\Delta\phi^A = X^{Aa}\lambda_a - \frac{1}{2}Y^A\lambda^2 = \left(s^a\phi^A\right)\lambda_a + \frac{1}{4}\left(s^2\phi^A\right)\lambda^2, \tag{3.7}$$

which implies that the finite variation $\Delta\phi^A$ includes the generators of BRST–antiBRST transformations (s^1, s^2), as well as their commutator $s^2 = \varepsilon_{ab}s^b s^a = s^1 s^2 - s^2 s^1$.

According to (2.24), (3.4), (3.7) and $\lambda_a\lambda^2 = \lambda^4 \equiv 0$, the variation $\Delta F(\phi)$ of an arbitrary functional $F(\phi)$ under the finite BRST–antiBRST transformations is given by

$$\begin{aligned}\Delta F &= F_{,A}X^{Aa}\lambda_a - \frac{1}{2}F_{,A}Y^A\lambda^2 + \frac{1}{2}F_{,AB}X^{Bb}\lambda_b X^{Aa}\lambda_a \\ &= \left(F_{,A}X^{Aa}\right)\lambda_a + \frac{1}{2}\left(\frac{1}{2}\varepsilon_{ab}F_{,BA}X^{Aa}X^{Bb}(-1)^{\varepsilon_B} - F_{,A}Y^A\right)\lambda^2 \\ &= \left(s^a F\right)\lambda_a + \frac{1}{4}\left(s^2 F\right)\lambda^2.\end{aligned} \tag{3.8}$$

This relation allows one to study the group properties of finite BRST–antiBRST transformations (3.7), with account taken for the fact that these transformations do not form a Lie superalgebra, nor a vector superspace structure, due to the presence of the term which is quadratic in λ_a. Namely, we have (for details, see Appendix A)

$$\Delta_{(1)}\Delta_{(2)}F = \left(s^a\Delta_{(2)}F\right)\lambda_{(1)a} + \frac{1}{4}\left(s^2\Delta_{(2)}F\right)\lambda^2_{(1)} \equiv \left(s^a F\right)\vartheta_{(1,2)a} + \frac{1}{4}\left(s^2 F\right)\theta_{(1,2)}, \tag{3.9}$$

for certain functionals $\vartheta^a_{(1,2)} = \vartheta^a_{(1,2)}(\phi)$ and $\theta_{(1,2)} = \theta_{(1,2)}(\phi)$, constructed explicitly in (A.7), (A.8) from the parameters of finite transformations, which are generally field-dependent, $\lambda^a_{(j)} = \lambda^a_{(j)}(\phi)$, for $j = 1, 2$. Therefore, the commutator of finite variations has the form

$$\begin{aligned}&[\Delta_{(1)}, \Delta_{(2)}]F = \left(s^a F\right)\vartheta_{[1,2]a} + \frac{1}{4}\left(s^2 F\right)\theta_{[1,2]}, \qquad \vartheta^a_{[1,2]} \equiv \vartheta^a_{(1,2)} - \vartheta^a_{(2,1)}, \\ &\theta_{[1,2]} \equiv \theta_{(1,2)} - \theta_{(2,1)},\end{aligned} \tag{3.10}$$

where $\vartheta^a_{[1,2]}$, $\theta_{[1,2]}$ are given explicitly by (A.11), (A.12) and possess the symmetry properties $\vartheta^a_{[1,2]} = -\vartheta^a_{[2,1]}$, $\theta_{[1,2]} = -\theta_{[2,1]}$. In particular, assuming $F(\phi) = \phi^A$ in (3.10), we have

$$[\Delta_{(1)}, \Delta_{(2)}]\phi^A = \left(s^a\phi^A\right)\vartheta_{[1,2]a} + \frac{1}{4}\left(s^2\phi^A\right)\theta_{[1,2]}. \tag{3.11}$$

In general, the commutator (3.11) of finite non-linear transformations (3.7) does not belong to the class of these transformations, due to the opposite symmetry properties of $\vartheta_{[1,2]a}\vartheta^a_{[1,2]}$ and $\theta_{[1,2]}$,

$$\vartheta_{[1,2]a}\vartheta^a_{[1,2]} = \vartheta_{[2,1]a}\vartheta^a_{[2,1]}, \qquad \theta_{[1,2]} = -\theta_{[2,1]}, \tag{3.12}$$

which reflects the fact that a finite BRST–antiBRST transformation looks as a group element, i.e., not as an element of a Lie superalgebra; however, the linear approximation $\Delta^{\mathrm{lin}}\phi^A = (s^a\phi^A)\lambda_a$ to a finite transformation $\Delta\phi^A = \Delta^{\mathrm{lin}}\phi^A + O(\lambda^2)$ does form an algebra. Indeed, due to (A.9), (A.11), (A.12), we have

$$\left[\Delta^{\mathrm{lin}}_{(1)}, \Delta^{\mathrm{lin}}_{(2)}\right]F = \Delta^{\mathrm{lin}}_{[1,2]}F = \left(s^a F\right)\lambda_{[1,2]a}, \qquad \lambda^a_{[1,2]} \equiv \left(s_b\lambda^a_{(1)}\right)\lambda^b_{(2)} - \left(s_b\lambda^a_{(2)}\right)\lambda^b_{(1)}. \tag{3.13}$$

[5] Finite BRST–antiBRST transformations (3.7) may be regarded as an extension of finite "mixed BRST" transformations [26], which include only the linear dependence on odd-valued parameters Θ_1, Θ_2; see Introduction for details.

Thus, the construction of finite BRST–antiBRST transformations (3.7) reduces to the usual BRST–antiBRST transformations (2.19), $\delta\phi^A = \Delta^{\text{lin}}\phi^A$, linear in the infinitesimal parameter $\mu_a = \lambda_a$, as one selects in (3.7) the approximation that forms an algebra with respect to the commutator.

Let us now consider the modification of the integration measure $d\phi \to d\phi'$ in (2.21) under the finite transformations $\phi^A \to \phi'^A = \phi^A + \Delta\phi^A$, with $\Delta\phi^A$ given by (3.7),

$$d\phi' = d\phi\,\text{Sdet}\left(\frac{\delta\phi'}{\delta\phi}\right),$$
$$\text{with Sdet}\left(\frac{\delta\phi'}{\delta\phi}\right) = \text{Sdet}(\mathbb{I}+M) = \exp\bigl[\text{Str}\ln(\mathbb{I}+M)\bigr] \equiv \exp(\Im), \tag{3.14}$$

where the Jacobian $\exp(\Im)$ has the form

$$\Im = \text{Str}\ln(\mathbb{I}+M) = -\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\text{Str}\bigl(M^n\bigr),$$
$$\text{for Str}\bigl(M^n\bigr) = \bigl(M^n\bigr)^A_A(-1)^{\varepsilon_A} \text{ and } M^A_B \equiv \frac{\delta(\Delta\phi^A)}{\delta\phi^B}. \tag{3.15}$$

In the case of *global* finite transformations, corresponding to $\lambda_a = \text{const}$, the integration measure remains invariant (for details, see Appendix B.1)

$$\Im(\phi) = 0 \quad \Longrightarrow \quad \left(\text{Sdet}\left(\frac{\delta\phi'}{\delta\phi}\right) = 1 \text{ and } d\phi' = d\phi\right). \tag{3.16}$$

Due to the invariance of the quantum action $S_F = S_0 + (1/2)s^a s_a F$ under $\phi^A \to \phi'^A$ the above implies that the integrand with the vanishing sources $\mathcal{I}_\phi \equiv d\phi\exp[(i/\hbar)S_F]$ in (2.21) is also invariant, $\mathcal{I}_{\phi'} = \mathcal{I}_\phi$, under the transformations (3.7), which justifies their interpretation as finite BRST–antiBRST transformations.

As we turn to finite *field-dependent* transformations, let us examine the particular case[6] $\lambda_a(\phi) = s_a\Lambda(\phi)$ with a certain even-valued potential, $\Lambda = \Lambda(\phi)$, which is inspired by infinitesimal field-dependent BRST–antiBRST transformations with the parameters (2.26). In this case, the integration measure takes the form (relation (3.18) is deduced in Appendix B.2)

$$\Im(\phi) = -2\ln\bigl[1+f(\phi)\bigr], \quad \text{with } f(\phi) = -\frac{1}{2}s^2\Lambda(\phi), \text{ for } s^a s_a = -s^2, \tag{3.17}$$
$$d\phi' = d\phi\exp\left[\frac{i}{\hbar}(-i\hbar\Im)\right] = d\phi\exp\left\{\frac{i}{\hbar}\left[i\hbar\,\ln\left(1+\frac{1}{2}s^a s_a\Lambda\right)^2\right]\right\}. \tag{3.18}$$

In view of the invariance of the quantum action $S_F(\phi)$ under (3.7), the change $\phi^A \to \phi'^A = \phi^A + \Delta\phi^A$ induces in (2.21) the following transformation of the integrand with the vanishing sources, $\mathcal{I}_\phi \equiv d\phi\exp[(i/\hbar)S_F(\phi)]$:

$$\mathcal{I}_{\phi+\Delta\phi} = d\phi\exp\bigl[\Im(\phi)\bigr]\exp\bigl[(i/\hbar)S_F(\phi+\Delta\phi)\bigr]$$
$$= d\phi\exp\bigl\{(i/\hbar)\bigl[S_F(\phi) - i\hbar\Im(\phi)\bigr]\bigr\}, \tag{3.19}$$

[6] Notice that the parameters λ_a, $a = 1, 2$, in the case $\lambda_a = s_a\Lambda$ are not functionally independent: $s^1\lambda_1 + s^2\lambda_2 = -s^2\Lambda$.

whence

$$\mathcal{I}_{\phi+\Delta\phi} = d\phi \exp\{(i/\hbar)[S_F(\phi) + i\hbar \ln(1 + s^a s_a \Lambda(\phi)/2)^2]\}. \tag{3.20}$$

Due to the explicit form of the initial quantum action $S_F = S_0 + (1/2)s^a s_a F$, the BRST–antiBRST-exact contribution $i\hbar \ln(1 + s^a s_a \Lambda/2)^2$ to the action S_F, resulting from the transformation of the integration measure, can be interpreted as a change of the gauge-fixing functional made in the original integrand $\mathcal{I}_\phi$,

$$i\hbar \ln(1 + s^a s_a \Lambda/2)^2 = s^a s_a (\Delta F/2) \tag{3.21}$$

$$\Longrightarrow \quad \mathcal{I}_{\phi+\Delta\phi} = d\phi \exp\{(i/\hbar)[S_0 + (1/2)s^a s_a (F + \Delta F)]\} = \mathcal{I}_\phi|_{F\to F+\Delta F}, \tag{3.22}$$

for a certain $\Delta F(\phi)$, whose relation to $\Lambda(\phi)$ is discussed below. In other words, the field-dependent transformations with the parameters $\lambda_a = s_a \Lambda$ amount to a *precise change of the gauge-fixing functional*. As a consequence, the integrand in (2.21) for $J_A = 0$, corresponding to the quantum action $S_{F+\Delta F} = S_0 + (1/2)s^a s_a (F + \Delta F)$ with a modified gauge-fixing functional, is invariant under both the infinitesimal, $\delta\phi^A$, and finite, $\Delta\phi^A$, BRST–antiBRST transformations, with constant parameters μ_a and λ_a in (2.19) and (3.7), respectively.

Let us denote by $T^{(\Delta F)}$ the operation that transforms an integrand $\mathcal{I}_\phi^{(F)}$ into $\mathcal{I}_\phi^{(F+\Delta F)}$, corresponding to the respective gauge-fixing functionals F and $F + \Delta F$,

$$T^{(\Delta F)}\colon \mathcal{I}_\phi^{(F)} \to \mathcal{I}_\phi^{(F+\Delta F)}, \tag{3.23}$$

which implies an additive composition law:

$$T^{(\Delta F_1)} \circ T^{(\Delta F_2)} = T^{(\Delta F_2)} \circ T^{(\Delta F_1)} = T^{(\Delta F_1+\Delta F_2)}. \tag{3.24}$$

As we denote by $\Lambda^{(\Delta F)}$ the gauge-fixing functional corresponding to ΔF, there follow the properties

$$\begin{aligned}&\ln(1 + s^a s_a \Lambda^{(\Delta F_1+\Delta F_2)}/2)^2 = \ln(1 + s^a s_a \Lambda^{(\Delta F_1)}/2)^2 + \ln(1 + s^a s_a \Lambda^{(\Delta F_2)}/2)^2,\\ &\Lambda^{(0)} = 0,\end{aligned} \tag{3.25}$$

implying relations between $s^2\Lambda^{(\Delta F_1+\Delta F_2)}$ and $s^2\Lambda^{(\Delta F_j)}$ for $j = 1, 2$, as well as between $s^2\Lambda^{(-\Delta F)}$ and $s^2\Lambda^{(\Delta F)}$:

$$s^2\Lambda^{(\Delta F_1+\Delta F_2)} = s^2(\Lambda^{(\Delta F_1)} + \Lambda^{(\Delta F_2)}) - (s^2\Lambda^{(\Delta F_1)})(s^2\Lambda^{(\Delta F_2)})/2, \tag{3.26}$$

$$s^2\Lambda^{(-\Delta F)} = -(s^2\Lambda^{(\Delta F)})[1 - (s^2\Lambda^{(\Delta F)})/2]^{-1}. \tag{3.27}$$

The relation (3.21) between the potential $\Lambda(\phi)$ and the variation $\Delta F(\phi)$ of the gauge-fixing functional can be considered as a compensation equation (for the unknown functional $\Delta F(\phi)$, with a given $\Lambda(\phi)$, and vice versa),

$$i\hbar \ln(1 + s^a s_a \Lambda(\phi)/2)^2 = s^a s_a \Delta F(\phi)/2, \tag{3.28}$$

whose solution, up to BRST–antiBRST-exact terms, has the form

$$\Delta F(\phi) = 2i\hbar \Lambda(\phi)(s^a s_a \Lambda(\phi))^{-1} \ln(1 + s^a s_a \Lambda(\phi)/2)^2. \tag{3.29}$$

The relation (3.28) can be inverted as an equation for $\Lambda(\phi)$, namely,

$$s^a s_a \Lambda = 2\left[\exp\left(\frac{1}{4i\hbar} s^a s_a \Delta F\right) - 1\right]. \tag{3.30}$$

Up to BRST–antiBRST-exact terms, its solution reads

$$
\begin{aligned}
\Lambda &= 2\Delta F\left(s^a s_a \Delta F\right)^{-1}\left[\exp\left(\frac{1}{4i\hbar}s^b s_b \Delta F\right) - 1\right] \\
&= \frac{1}{2i\hbar}\Delta F \sum_{n=0}^{\infty} \frac{1}{(n+1)!}\left(\frac{1}{4i\hbar}s^a s_a \Delta F\right)^n,
\end{aligned} \tag{3.31}
$$

whence

$$
\begin{aligned}
\lambda_a = s_a \Lambda &= \frac{1}{2i\hbar}(s_a \Delta F)\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\left(\frac{1}{4i\hbar}s^b s_b \Delta F\right)^n \\
&= \frac{1}{2i\hbar}(s_a \Delta F)\left[1 + \frac{1}{2!}\left(\frac{1}{4i\hbar}s^b s_b \Delta F\right)\right. \\
&\quad \left. + \frac{1}{3!}\left(\frac{1}{4i\hbar}s^b s_b \Delta F\right)^2 + \frac{1}{4!}\left(\frac{1}{4i\hbar}s^b s_b \Delta F\right)^3 + \ldots\right].
\end{aligned} \tag{3.32}
$$

In particular, the first order of $\lambda_a = \mu_a$ in powers of ΔF has the form

$$
\mu_a = -\frac{i}{2\hbar}(s_a \Delta F). \tag{3.33}
$$

Using (3.32), one can construct a finite BRST–antiBRST transformation that connects two quantum theories of Yang–Mills type corresponding to some gauge-fixing functionals F and $F + \Delta F$ for a given finite variation ΔF. The symmetry of the integrand in (2.21) for $J_A = 0$ under the transformations (3.7) allows one to establish the independence of the S-matrix from the choice of a gauge. Indeed, suppose $Z_F \equiv Z(0)$ and change the gauge $F \to F + \Delta F$ by a finite value ΔF. In the functional integral for $Z_{F+\Delta F}$ we now make the change of variables (3.7). Then, selecting the parameters $\lambda_a = s_a \Lambda$ to meet the condition

$$
i\hbar \ln\left(1 + s^a s_a \Lambda/2\right)^2 = -(1/2)s^a s_a \Delta F, \tag{3.34}
$$

cf. (3.28), we find that $Z_{F+\Delta F} = Z_F$, whence, due to the equivalence theorem [41], the S-matrix is gauge-independent. In the particular case of an infinitesimal variation ΔF, condition (3.34) produces, in virtue of (3.33), precisely the form (2.26) of field-dependent parameters $\lambda_a = \mu_a$ in the framework of infinitesimal BRST–antiBRST transformations.

As we identify $\lambda_a = s_a \Lambda$ with a solution of (3.28), $\Lambda^{(\Delta F)} \equiv \Lambda(\Delta F)$, the representation (2.21) describes the dependence of the functional $Z_F(J)$ on a finite variation of the gauge:

$$
\begin{aligned}
\Delta Z_F(J) = \frac{i}{\hbar} Z_F(J) \Big\langle J_A \Big[&\left(s^a \phi^A\right) s_a \Lambda(-\Delta F) + \frac{1}{4}\left(s^2 \phi^A\right)\left[s\Lambda(-\Delta F)\right]^2 \\
&+ \frac{i}{4\hbar}\varepsilon_{ab}\left(s^a \phi^A\right) J_B \left(s^b \phi^B\right)\left[s\Lambda(-\Delta F)\right]^2 \Big]\Big\rangle_{F,J},
\end{aligned} \tag{3.35}
$$

where $\Delta Z_F(J) \equiv Z_{F+\Delta F}(J) - Z_F(J)$. The above relation (3.35) generalizes the gauge-dependence of $Z(J)$ in Yang–Mills type theories to the case of finite variations of the gauge.

4. Correspondence between gauges in Yang–Mills theories

In this section, we consider the Yang–Mills theory, given by the action

$$S_0(A) = -\frac{1}{4}\int d^D x\, F^m_{\mu\nu}F^{m\mu\nu}, \quad \text{for } F^m_{\mu\nu} = \partial_\mu A^m_\nu - \partial_\nu A^m_\mu + f^{mnl}A^n_\mu A^l_\nu, \tag{4.1}$$

with the Lorentz indices $\mu, \nu = 0, 1, \dots, D-1$, the metric tensor $\eta_{\mu\nu} = \mathrm{diag}(-, +, \dots, +)$, and the totally antisymmetric $su(N)$ structure constants f^{lmn} for $l, m, n = 1, \dots, N^2 - 1$.

The action (4.1) is invariant under the gauge transformations

$$\delta A^m_\mu(x) = D^{mn}_\mu(x)\zeta^n(x) = \int d^D y\, R^{mn}_\mu(x; y)\zeta^n(y), \quad D^{mn}_\mu = \delta^{mn}\partial_\mu + f^{mln}A^l_\mu, \tag{4.2}$$

with arbitrary Bosonic functions $\zeta^n(y)$ in $\mathbb{R}^{1,D-1}$, the covariant derivative D^{mn}_μ, and the generators $R^{mn}_\mu(x; y) = R^i_\alpha$ of the gauge transformations, the condensed indices being $i = (\mu, m, x)$, $\alpha = (n, y)$. The generators R^i_α in (4.2) form a closed gauge algebra with $M^{ij}_{\alpha\beta} = 0$ in (2.3), whereas the structure coefficients $F^\gamma_{\alpha\beta}$ arising in (2.3) are given by

$$F^\gamma_{\alpha\beta} = f^{lmn}\delta(x - z)\delta(y - z), \quad \text{for } \alpha = (m, x),\ \beta = (n, y),\ \gamma = (l, z). \tag{4.3}$$

The total configuration space of fields ϕ^A and the corresponding antifields ϕ^*_{Aa}, $\bar{\phi}_A$ of the theory are given by

$$\phi^A = \left(A^{\mu m}, B^m, C^{ma}\right), \quad \phi^*_{Aa} = \left(A^{*m}_{\mu a}, B^{*m}_a, C^{*m}_{ab}\right), \quad \bar{\phi}_A = \left(\bar{A}^m_\mu, \bar{B}^m, \bar{C}^m_a\right). \tag{4.4}$$

With allowance made for (2.1), the Grassmann parity and ghost number assume the values

$$\varepsilon\left(\phi^A\right) \equiv (0, 0, 1), \qquad \mathrm{gh}\left(\phi^A\right) = \left(0, 0, (-1)^{a+1}\right). \tag{4.5}$$

The generating equations (2.5) with the boundary condition $S|_{\phi^* = \bar{\phi} = 0} = S_0$ are solved by a functional linear in the antifields (for details, see (C.3), (C.4) in Appendix C)

$$S = S_0 + \int d^D x \left(A^{*m}_{\mu a}X^{\mu ma}_1 + B^{*m}_a X^{ma}_2 + C^{*m}_{ab}X^{mab}_3 + \bar{A}^m_\mu Y^{\mu m}_1 + \bar{C}^m_a Y^{ma}_3\right), \tag{4.6}$$

where the functionals $X^{Aa} = \delta S/\delta\phi^*_{Aa} = (X^{\mu ma}_1, X^{ma}_2, X^{mab}_3)$ and $Y^A = \delta S/\delta\bar{\phi}_A = (Y^{\mu m}_1, Y^m_2, Y^{ma}_3)$ are given by

$$\begin{aligned}
&X^{\mu ma}_1 = D^{\mu mn}C^{na}, \qquad Y^{\mu m}_1 = D^{\mu mn}B^n + \frac{1}{2}f^{mnl}C^{la}D^{\mu nk}C^{kb}\varepsilon_{ba},\\
&X^{ma}_2 = -\frac{1}{2}f^{mnl}B^l C^{na} - \frac{1}{12}f^{mnl}f^{lrs}C^{sb}C^{ra}C^{nc}\varepsilon_{cb}, \qquad Y^m_2 = 0,\\
&X^{mab}_3 = -\varepsilon^{ab}B^m - \frac{1}{2}f^{mnl}C^{lb}C^{na},\\
&Y^{ma}_3 = f^{mnl}B^l C^{na} + \frac{1}{6}f^{mnl}f^{lrs}C^{sb}C^{ra}C^{nc}\varepsilon_{cb}.
\end{aligned} \tag{4.7}$$

Hence, the finite BRST–antiBRST transformations $\Delta\phi^A = X^{Aa}\lambda_a - (1/2)Y^A\lambda^2$ read as follows:

$$\Delta A^m_\mu = D^{mn}_\mu C^{na}\lambda_a - \frac{1}{2}\left(D^{mn}_\mu B^n + \frac{1}{2}f^{mnl}C^{la}D^{nk}_\mu C^{kb}\varepsilon_{ba}\right)\lambda^2, \tag{4.8}$$

$$\Delta B^m = -\frac{1}{2}\left(f^{mnl}B^l C^{na} + \frac{1}{6}f^{mnl}f^{lrs}C^{sb}C^{ra}C^{nc}\varepsilon_{cb}\right)\lambda_a, \tag{4.9}$$

$$\Delta C^{ma} = \left(\varepsilon^{ab}B^m - \frac{1}{2}f^{mnl}C^{la}C^{nb}\right)\lambda_b - \frac{1}{2}\left(f^{mnl}B^l C^{na} + \frac{1}{6}f^{mnl}f^{lrs}C^{sb}C^{ra}C^{nc}\varepsilon_{cb}\right)\lambda^2, \tag{4.10}$$

where the approximation linear in $\lambda_a = \mu_a$ produces the infinitesimal BRST–antiBRST transformations $\delta\phi^A = X^{Aa}\mu_a = (s^a\phi^A)\mu_a$.

To construct the generating functional of Green's functions $Z(J)$ in (2.21), we choose the gauge functional $F = F(\phi)$ to be diagonal in $A^{\mu m}$, C^{ma}, namely,

$$F(A, C) = -\frac{1}{2}\int d^Dx\left(\alpha A^m_\mu A^{m\mu} + \beta\varepsilon_{ab}C^{ma}C^{mb}\right). \tag{4.11}$$

The quantum action $S_F(\phi)$ corresponding to this gauge-fixing functional reads (see (C.5)–(C.22) in Appendix C)

$$S_F(A, B, C) = S_0(A) + (1/2)s^a s_a F(A, C) = S_0(A) + S_{\mathrm{gf}}(A, B) + S_{\mathrm{gh}}(A, C) + S_{\mathrm{add}}(C), \tag{4.12}$$

where the gauge-fixing term S_{gf}, the ghost term S_{gh}, and the interaction term S_{add}, quartic in C^{ma}, are given by

$$S_{\mathrm{gf}} = \int d^Dx\left[\alpha\left(\partial^\mu A^m_\mu\right) - \beta B^m\right]B^m, \qquad S_{\mathrm{gh}} = \frac{\alpha}{2}\int d^Dx\left(\partial^\mu C^{ma}\right)D^{mn}_\mu C^{nb}\varepsilon_{ab}, \tag{4.13}$$

$$S_{\mathrm{add}} = \frac{\beta}{24}\int d^Dx\, f^{mnl}f^{lrs}C^{sa}C^{rc}C^{nb}C^{md}\varepsilon_{ab}\varepsilon_{cd}. \tag{4.14}$$

Let us examine the choice of the coefficients α, β leading to R_ξ-like gauges. Namely, in view of the contribution S_{gf} to the quantum action S_F,

$$S_{\mathrm{gf}} = \int d^Dx\left[\alpha\left(\partial^\mu A^m_\mu\right) - \beta B^m\right]B^m, \tag{4.15}$$

we impose the conditions

$$\alpha = 1, \qquad \beta = -\frac{\xi}{2}. \tag{4.16}$$

Thus, the gauge-fixing functional $F_{(\xi)} = F_{(\xi)}(A, C)$ corresponding to an R_ξ-like gauge can be chosen as

$$F_{(\xi)} = \frac{1}{2}\int d^Dx\left(-A^m_\mu A^{m\mu} + \frac{\xi}{2}\varepsilon_{ab}C^{ma}C^{mb}\right), \quad \text{so that} \tag{4.17}$$

$$F_{(0)} = -\frac{1}{2}\int d^Dx\, A^m_\mu A^{m\mu} \quad \text{and} \quad F_{(1)} = \frac{1}{2}\int d^Dx\left(-A^m_\mu A^{m\mu} + \frac{1}{2}\varepsilon_{ab}C^{ma}C^{mb}\right), \tag{4.18}$$

where the gauge-fixing functional $F_{(0)}(A)$ induces the contribution $S_{\mathrm{gf}}(A, B)$ to the quantum action that arises in the case of the Landau gauge $\chi(A) = \partial^\mu A^m_\mu = 0$ for $(\alpha, \beta) = (1, 0)$ in (4.15), whereas the functional $F_{(1)}(A, C)$ corresponds to the Feynman (covariant) gauge $\chi(A, B) = \partial^\mu A^m_\mu + (1/2)B^m = 0$ for $(\alpha, \beta) = (1, -1/2)$ in (4.15).

Let us find the parameters $\lambda_a = s_a \Lambda$ of a finite field-dependent BRST–antiBRST transformation that connects an R_ξ gauge with an $R_{\xi+\Delta\xi}$ gauge, according to (3.32), where

$$\Delta F_{(\xi)} = F_{(\xi+\Delta\xi)} - F_{(\xi)} = \frac{\Delta\xi}{4}\varepsilon_{ab}\int d^D x\, C^{ma}C^{mb}. \tag{4.19}$$

Explicitly,

$$\delta(\Delta F_{(\xi)}) = s^a(\Delta F_{(\xi)})\mu_a = \frac{\Delta\xi}{2}\varepsilon_{ba}\int d^D x\, C^{mb}\delta C^{ma}, \tag{4.20}$$

where $\delta C^{ma} = (\varepsilon^{ab}B^m - (1/2)f^{mnl}C^{la}C^{nb})\mu_b$ is the linear part of the finite BRST–antiBRST transformation (4.10), which implies

$$s^a(\Delta F_{(\xi)}) = \frac{\Delta\xi}{2}\varepsilon_{bc}\int d^D x\, C^{mb}\left(\varepsilon^{ca}B^m - \frac{1}{2}f^{mnl}C^{lc}C^{na}\right). \tag{4.21}$$

In order to calculate $s^a s_a(\Delta F_{(\xi)})$, we remind that

$$\begin{aligned}\frac{1}{2}s^a s_a F_{(\xi)} &= S_{\mathrm{gf}} + S_{\mathrm{gh}} + S_{\mathrm{add}}|_{\alpha=1,\beta=-\xi/2}\\ &= \int d^D x\left\{\left[\left(\partial^\mu A^m_\mu\right) + \frac{\xi}{2}B^m\right]B^m + \frac{1}{2}\left(\partial^\mu C^{ma}\right)D^{mn}_\mu C^{nb}\varepsilon_{ab}\right.\\ &\quad\left. - \frac{\xi}{48}f^{mnl}f^{lrs}C^{sa}C^{rc}C^{nb}C^{md}\varepsilon_{ab}\varepsilon_{cd}\right\},\end{aligned} \tag{4.22}$$

whence

$$s^a s_a(\Delta F_{(\xi)}) = \Delta\xi\int d^D x\left(B^m B^m - \frac{1}{24}f^{mnl}f^{lrs}C^{sa}C^{rc}C^{nb}C^{md}\varepsilon_{ab}\varepsilon_{cd}\right). \tag{4.23}$$

Finally, the functionals $\lambda_a(\phi)$ that connect an R_ξ-like gauge to an $R_{\xi+\Delta\xi}$-like gauge are given by (3.32)

$$\begin{aligned}\lambda_a &= \frac{\Delta\xi}{4i\hbar}\varepsilon_{ab}\int d^D x\left(B^n C^{nb}\right)\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\\ &\quad\times\left[\frac{1}{4i\hbar}\Delta\xi\int d^D y\left(B^u B^u - \frac{1}{24}f^{uwt}f^{trs}C^{sc}C^{rp}C^{wd}C^{uq}\varepsilon_{cd}\varepsilon_{pq}\right)\right]^n.\end{aligned} \tag{4.24}$$

In particular, the first order of $\lambda_a = \mu_a$ in powers of $\Delta F_{(\xi)}$ has the form (3.33)

$$\mu_a = -\frac{i}{2\hbar}s_a\Delta F_{(\xi)} = -\frac{i\Delta\xi}{4\hbar}\varepsilon_{ab}\int d^D x\, B^m C^{mb}. \tag{4.25}$$

We have thus solved the problem of reaching any gauge in the family of R_ξ-like gauges, starting from a certain gauge encoded in the path integral by a functional $F_{(\xi)}$, within the framework of BRST–antiBRST quantization for Yang–Mills theories by means of finite BRST–antiBRST transformations with field-dependent parameters λ_a in (4.24). Generally, if the BRST–antiBRST invariant quantum action S_{F_0} of a Yang–Mills theory is given in terms of a gauge induced by a gauge-fixing functional F_0, then, in order to reach the quantum action S_F in terms of another gauge induced by a gauge-fixing functional F, it is sufficient to make a change of variables in the path integral (2.21) with S_{F_0}, given by a finite field-dependent BRST–antiBRST transformation with an Sp(2)-doublet of the odd-valued functionals

$$\lambda_a(F-F_0)=\frac{1}{2i\hbar}\bigl[s_a(F-F_0)\bigr]\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\left(\frac{1}{4i\hbar}s^b s_b(F-F_0)\right)^n. \tag{4.26}$$

In particular, if we choose $F_0=F_{(\xi)}$, with $F_{(\xi)}$ given by (4.17), then the above relation (4.26) describes the transition from an R_ξ-like gauge to a gauge parameterized by an arbitrary gauge-fixing functional $F=F(A,B,C)$.

5. Gribov–Zwanziger action in R_ξ-like gauges

Let us extend the construction of the Gribov horizon [33] to the case of a BRST–antiBRST invariant Yang–Mills theory in a way consistent with the gauge-independence of the S-matrix. To this end, we examine the sum of the Yang–Mills quantum action (4.12) in the Landau gauge $\partial^\mu A^m_\mu=0$ (with the gauge-fixing functional $F_{(0)}$ in (4.18) corresponding to the case $\alpha=1$, $\beta=0$) and the non-local horizon functional [34]

$$h(A)=\gamma^2\int d^Dx\,d^Dy\,f^{mrl}A^r_\mu(x)\bigl(K^{-1}\bigr)^{mn}(x;y)f^{nsl}A^{\mu s}(y)+\gamma^2 D\bigl(N^2-1\bigr), \tag{5.1}$$

where K^{-1} is the inverse,

$$\begin{aligned}\int d^Dz\bigl(K^{-1}\bigr)^{ml}(x;z)(K)^{ln}(z;y)&=\int d^Dz\bigl(K^{-1}\bigr)^{nl}(x;z)(K)^{lm}(z;y)\\&=\delta^{mn}\delta(x-y),\end{aligned} \tag{5.2}$$

of the Faddeev–Popov operator K induced by the gauge-fixing functional $F_{(\xi\to 0)}$ corresponding to the Landau gauge $\partial^\mu A^m_\mu=0$ in the BRST approach,

$$K^{mn}(x;y)=\bigl(\delta^{mn}\partial^2+f^{mln}A^l_\mu\partial^\mu\bigr)\delta(x-y),\qquad K^{mn}(x;y)=K^{nm}(y;x), \tag{5.3}$$

whereas $\gamma\in\mathbb{R}$ is the so-called thermodynamic, or Gribov, parameter [34], introduced in a self-consistent way by the gap equation for an analogue S_h of the Gribov–Zwanziger action in the BRST–antiBRST approach:

$$\frac{\partial}{\partial\gamma}\left\{\frac{\hbar}{i}\ln\left[\int D\phi\exp\left(\frac{i}{\hbar}S_h\right)\right]\right\}=\frac{\partial\mathcal{E}_{\rm vac}}{\partial\gamma}=0. \tag{5.4}$$

In (5.4), we have used the definition of the vacuum energy $\mathcal{E}_{\rm vac}$ and introduced a modified quantum action for the Gribov–Zwanziger model as an additive extension of the Yang–Mills quantum action S_{F_0} (4.12) in the Landau gauge:

$$S_h(\phi)=S_{F_0}(\phi)+h(\phi),\qquad F_0=F_{(0)}. \tag{5.5}$$

The action $S_h(\phi)$ is not invariant under the finite BRST–antiBRST transformations:

$$\Delta S_h=\Delta h=\bigl(s^a h\bigr)\lambda_a+\frac{1}{4}\bigl(s^2h\bigr)\lambda^2\neq 0, \tag{5.6}$$

indeed, according to $\Delta\phi^A=(s^a\phi^A)\lambda_a+(1/4)(s^2\phi^A)\lambda^2$, with allowance for (4.8)–(4.10), (A.2), we have

$$
\begin{aligned}
s^a h = {} & \gamma^2 f^{mrk} f^{kns} \int d^D x\, d^D y \Big[2 D^{rl}_{\mu} C^{la}(x) \big(K^{-1}\big)^{mn}(x;y) \\
& - f^{utv} \int d^D x'\, d^D y' A^r_{\mu}(x) \big(K^{-1}\big)^{mu}\big(x;x'\big) K^{tl}\big(x';y'\big) C^{la}\big(y'\big) \big(K^{-1}\big)^{vn}\big(y';y\big) \Big] \\
& \times A^{s\mu}(y)
\end{aligned}
\tag{5.7}
$$

and

$$
\begin{aligned}
s^2 h = {} & \gamma^2 f^{mrk} f^{kns} \int d^D x\, d^D y \\
& \times \bigg\{ 4 \Big(-D^{rt}_{\mu} B^t + \frac{1}{2} f^{rtl} C^{la} D^{tu}_{\mu} C^{ub} \varepsilon_{ab} \Big)(x) \big(K^{-1}\big)^{mn}(x;y) A^{s\mu}(y) \\
& + 2\varepsilon_{ab} D^{rl}_{\mu} C^{la}(x) \big(K^{-1}\big)^{mn}(x;y) D^{st\mu} C^{tb}(y) \\
& - 4\varepsilon_{ab} f^{utv} \int d^D x'\, d^D y'\, D^{rl}_{\mu} C^{la}(x) \big(K^{-1}\big)^{mu}\big(x;x'\big) K^{tw}\big(x';y'\big) \\
& \times C^{wb}\big(y'\big) \big(K^{-1}\big)^{vn}\big(y';y\big) A^{s\mu}(y) + f^{utv} \int d^D x'\, d^D y'\, A^r_{\mu}(x) \\
& \times \Big[-\varepsilon_{ab} f^{u't'v'} \int d^D x''\, d^D y'' \big(K^{-1}\big)^{mu'}\big(x;x''\big) K^{t'l'}\big(x'';y''\big) C^{l'a}\big(y''\big) \\
& \times \big(K^{-1}\big)^{v'u}\big(y'';x'\big) K^{tl}\big(x';y'\big) C^{lb}\big(y'\big) \big(K^{-1}\big)^{vn}\big(y';y\big) \\
& - \varepsilon_{ab} f^{tlt'} \big(K^{-1}\big)^{mu}\big(x;x'\big) K^{t'l'}\big(x';y'\big) C^{l'a}\big(y'\big) C^{lb}\big(x'\big) \big(K^{-1}\big)^{vn}\big(y';y\big) \\
& + 2 \big(K^{-1}\big)^{mu}\big(x;x'\big) K^{tl}\big(x';y'\big) B^l\big(y'\big) \big(K^{-1}\big)^{vn}\big(y';y\big) \\
& + \varepsilon_{ab} f^{u't'v'} \big(K^{-1}\big)^{mu}\big(x;x'\big) K^{tl}\big(x';y'\big) C^{la}\big(y'\big) \\
& \times \int d^D x''\, d^D y'' \big(K^{-1}\big)^{vu'}\big(y';x''\big) K^{t'l'}\big(x'';y''\big) C^{l'b}\big(y''\big) \big(K^{-1}\big)^{v'n}\big(y'';y\big) \Big] \\
& \times A^{s\mu}(y) \bigg\},
\end{aligned}
\tag{5.8}
$$

where we have used the identity

$$
s^a K^{mn}(x;y) = f^{mrn} K^{rs}(x;y) C^{sa}(y).
\tag{5.9}
$$

To determine the horizon functional for a general R_ξ-like gauge in the BRST–antiBRST description, we propose

$$
\begin{aligned}
h_\xi = {} & h + \frac{1}{2i\hbar} \big(s^a h\big) (s_a \Delta F_{(\xi)}) \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \Big(\frac{1}{4i\hbar} s^b s_b \Delta F_{(\xi)} \Big)^n \\
& - \frac{1}{16\hbar^2} \big(s^2 h\big) (s \Delta F_{(\xi)})^2 \bigg[\sum_{n=0}^{\infty} \frac{1}{(n+1)!} \Big(\frac{1}{4i\hbar} s^b s_b \Delta F_{(\xi)} \Big)^n \bigg]^2 .
\end{aligned}
\tag{5.10}
$$

Here, $s^a h$ and $s^2 h$ are given by (5.7), (5.8), while $s_a \Delta F_{(\xi)}$ and $s^a s_a \Delta F_{(\xi)}$ are given by (4.21), (4.23) for $\Delta\xi = \xi$, whereas the Sp(2)-doublet $\lambda^a_\xi(\phi)$ of field-dependent anticommuting parameters in (4.24) relates the Landau gauge to an arbitrary R_ξ-like gauge:

$$\Delta F_{(\xi)} = F_{(\xi)} - F_{(0)} = \frac{\xi}{4}\varepsilon_{ab}\int d^D x\, C^{ma}C^{mb}, \tag{5.11}$$

$$s_a \Delta F_{(\xi)} = \frac{\xi}{2}\varepsilon_{ab}\int d^D x\, B^m C^{mb}, \tag{5.12}$$

$$s^a s_a \Delta F_{(\xi)} = \xi\int d^D x\biggl(B^m B^m - \frac{1}{24} f^{mnl} f^{lrs} C^{sa}C^{rc}C^{nb}C^{md}\varepsilon_{ab}\varepsilon_{cd}\biggr). \tag{5.13}$$

In particular, the approximation linear in ξ implies, $\lambda^a_\xi(\phi) = s^a \Lambda_\xi(\phi)$ for $\Lambda_\xi(\phi) = \frac{\xi}{8i\hbar}\times \varepsilon_{ab}\int d^D x\, C^{ma}C^{mb}$,

$$\begin{aligned} h_\xi(\phi) = h(A) &+ \frac{\xi}{4i\hbar}\varepsilon_{ab}\gamma^2 f^{mrl} f^{lns}\int d^D x\, d^D y \Bigl[2D^{rk}_\mu C^{ka}(x)\bigl(K^{-1}\bigr)^{mn}(x;y) \\ &- f^{m'l'n'}\int d^D x'\, d^D y'\, A^r_\mu(x)\bigl(K^{-1}\bigr)^{mm'}\bigl(x;x'\bigr)K^{l't'}\bigl(x';y'\bigr) \\ &\times C^{t'a}\bigl(y'\bigr)\bigl(K^{-1}\bigr)^{n'n}\bigl(y';y\bigr)\Bigr]A^{s\mu}(y)\int d^D z\bigl(B^w C^{wb}\bigr). \end{aligned} \tag{5.14}$$

Notice that even the approximation to $h_\xi(\phi)$ being linear in powers of ξ is different from the proposal [37] for the horizon functional given by R_ξ-gauges in terms of field-dependent BRST transformations, which reflects the Sp(2)-symmetric character of the dependence of $h_\xi(\phi)$ on the ghost and antighost fields C^{ma}.

The proposal (5.10) for the Gribov horizon functional in a general R_ξ-gauge is consistent with the study of gauge-independence for the generating functional of Green's functions, determined for a BRST–antiBRST extension of the Gribov–Zwanziger model as follows:

$$Z_{\mathrm{GZ},F_0}(J) = \int d\phi \exp\biggl\{\frac{i}{\hbar}\bigl[S_h(\phi) + J_A\phi^A\bigr]\biggr\}. \tag{5.15}$$

Indeed, making in the path integral for $Z_{\mathrm{GZ},F_0}(J)$ a change of variables being a finite field-dependent BRST–antiBRST transformation with the parameters $\lambda^a_\xi(\phi)$ given by (4.24), where $\Delta\xi = \xi$, we find, due to the fact that the Yang–Mills quantum action $S_{F_0}(\phi)$ transforms to $S_{F_\xi}(\phi)$, with $F_\xi = F_{(\xi)}$,

$$Z_{\mathrm{GZ},F_0}(J) = \int d\phi \exp\biggl\{\frac{i}{\hbar}\bigl[S_{F_\xi}(\phi) + h_\xi(\phi) + J_A\phi^A + J_A\Delta\phi^A\bigr]\biggr\}, \tag{5.16}$$

where $h_\xi(\phi)$ in (5.10) corresponds to an R_ξ-gauge. As a result, we have

$$\begin{aligned} &Z_{\mathrm{GZ},F_0}(J) \\ &\quad = Z_{\mathrm{GZ},F_\xi}(J)\biggl[1 + \frac{i}{\hbar}J_A\bigl\langle\bigl(s^a\phi^A\bigr)s_a\Lambda(\Delta F_{(\xi)})\bigr\rangle_{F_0,J} \\ &\qquad + \frac{i}{4\hbar}J_A\biggl\langle\bigl(s^2\phi^A\bigr)\bigl[s\Lambda(\Delta F_{(\xi)})\bigr]^2 + \frac{i}{\hbar}\varepsilon_{ab}\bigl(s^a\phi^A\bigr)J_B\bigl(s^b\phi^B\bigr)\bigl[s\Lambda(\Delta F_{(\xi)})\bigr]^2\biggr\rangle_{F_0,J}\biggr], \end{aligned} \tag{5.17}$$

where the vacuum expectation value is computed with respect to $Z_{\mathrm{GZ},F}(J)$. The relation (5.17) implies that neither the functional $Z_{\mathrm{GZ},F_\xi}(J)$ nor the S-matrix depends on the gauge (parameter ξ) at the extremals given by $J_A = 0$. This justifies our proposal for the horizon functional in the form[7] (5.10). At the same time, we note that the Gribov–Zwanziger model in BRST–antiBRST quantization encounters the problem of unitarity, since the gauge degrees of freedom, being non-dynamical in the Yang–Mills theory, should now be regarded as dynamical ones, due to the explicit form of the horizon functional $h_\xi(\phi)$.

Finally, it is possible to construct a Gribov horizon functional $h_F(\phi)$ in any differential gauge[8] induced by a gauge-fixing functional $F(\phi)$, starting from the horizon functional $h(A)$ in the Landau gauge, corresponding to the gauge-fixing functional $F_0(A)$. To this end, it is sufficient to make a change of variables in the path integral (5.15), given by a finite field-dependent BRST–antiBRST transformation with the Sp(2)-doublet $\lambda_a(F-F_0)$ of odd-valued functionals given by (4.26). Thus, the functional $h_F(\phi)$ reads as follows:

$$\begin{aligned} h_F = h &+ \frac{1}{2i\hbar}\big(s^a h\big)\big[s_a(F-F_0)\big]\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\bigg(\frac{1}{4i\hbar}s^b s_b(F-F_0)\bigg)^n \\ &- \frac{1}{16\hbar^2}\big(s^2 h\big)\big[s(F-F_0)\big]^2\bigg[\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\bigg(\frac{1}{4i\hbar}s^b s_b(F-F_0)\bigg)^n\bigg]^2. \end{aligned} \tag{5.18}$$

Generally, a finite change $F \to F + \Delta F$ of the gauge condition induces a finite change of any functional $G_F(\phi)$, so that in the reference frame corresponding to the gauge $F + \Delta F$ it can be represented according to (3.8), (4.26),

$$G_{F+\Delta F} = G_F + \big(s^a G_F\big)\lambda_a(\Delta F) + \frac{1}{4}\big(s^2 G_F\big)\lambda_a(\Delta F)\lambda^a(\Delta F), \tag{5.19}$$

which is an extension of the infinitesimal change $G_F \to G_F + \delta G_F$ induced by a variation of the gauge, $F \to F + \delta F$,

$$G_{F+\delta F} = G_F - \frac{i}{2\hbar}\big(s^a G_F\big)(s_a \delta F), \tag{5.20}$$

corresponding, in the case $G_F(A)$, to the gauge transformations (4.2), with the functions $\zeta^m(x)$ given below

$$\begin{aligned} \delta G_F &= G_{F+\delta F} - G_F \\ &= \int d^D x \frac{\delta G_F}{\delta A^{\mu m}(x)} D^{mn\mu}\zeta^n(x), \quad \text{where } \zeta^m(x) = -\frac{i}{2\hbar}C^{ma}(x)(s_a\delta F). \end{aligned} \tag{5.21}$$

Due to the presence of the term with $s^2 G_F$ in a finite gauge variation of a functional $G_F(A)$ depending only on the classical fields $A^{m\mu}$, the representation (5.19) is more general than the one that would correspond to the usual Lagrangian BRST approach (see relation (17) in [39]), having the form similar to (5.21), and thus also to (5.20).

We emphasize that the suggested method of using the finite field-dependent BRST–antiBRST transformations with the purpose of finding the Gribov–Zwanziger horizon functional in any differential gauge, starting from the Gribov–Zwanziger theory in the Landau gauge, is valid in

[7] There exist other ways to obtain the Gribov horizon functional h_ξ for gauges beyond the Landau gauge, see, e.g., [35,38]; however, in view of its non-perturbative character [34], the derivation procedure faces the problem of gauge dependence.

[8] Due to the result of Singer [42], Gribov copies should arise in non-Abelian gauge theories in case a differential gauge is used to fix the gauge ambiguity.

perturbation theory and preserves the number of physical degrees of freedom, without entering into contradiction with the result of [24] in the BRST setting of the problem. However, it is impossible to solve this problem (in particular, in the Yang–Mills theory) in terms of finite field-dependent BRST–antiBRST transformations [26], in view of the absence of a term being quadratic in powers of the odd-valued parameters, since the corresponding Yang–Mills quantum action fails to be BRST–antiBRST invariant, and the Jacobian of the corresponding change of variables with odd-valued functionally-dependent parameters does not generate terms which are entirely BRST–antiBRST-exact. These terms change the BRST–antiBRST-exact part of the action, as well as the extremals; however, they do not affect the number of physical degrees of freedom.

6. Discussion

In the present work, we have proposed the concept of finite BRST–antiBRST transformations for Yang–Mills theories in the Sp(2)-covariant Lagrangian quantization [15,16], realized in the form (3.5), (3.7), being polynomial in powers of a constant Sp(2)-doublet of anticommuting Grassmann parameters λ_a and leaving the quantum action of the Yang–Mills theory invariant to all orders in λ_a. In the case of constant λ_a, the set of finite BRST–antiBRST transformations forms an Abelian two-parametric Lie supergroup with the elements $g(\lambda) = \exp(\overleftarrow{s}^a \lambda_a) = (1 + \overleftarrow{s}^a \lambda_a + \frac{1}{4}\overleftarrow{s}^a \overleftarrow{s}_a \lambda^2)$, so that $\Delta\phi^A = \phi^A[\exp(\overleftarrow{s}^a \lambda_a) - 1]$, where $G\overleftarrow{s}^a \equiv s^a G$, for any $G = G(\phi)$. Secondly, this ensures exact invariance of the integrand in the generating functional of Green's functions $Z_F(J)$ with vanishing external sources J_A and also allows one to obtain the Ward identities.

We have determined the finite field-dependent BRST–antiBRST transformations as polynomials in the Sp(2)-doublet of Grassmann-odd functionals $\lambda_a(\phi)$, depending on the whole set of fields that compose the configuration space of Yang–Mills theories, and have also calculated the Jacobian (3.18) corresponding to this change of variables by using a special class of transformations with s_a-potential parameters $\lambda_a(\phi) = s_a \Lambda(\phi)$ for a Grassmann-even functional $\Lambda(\phi)$ and Grassmann-odd generators s_a of BRST–antiBRST transformations.

In comparison with finite field-dependent BRST transformations in Yang–Mills theories [23], in which a change of the gauge corresponds to a unique field-dependent parameter (up to BRST-exact terms), it is only functionally-dependent finite field-dependent BRST–antiBRST transformations with $\lambda_a = s_a \Lambda(\Delta F)$ that are in one-to-one correspondence with ΔF. We have found (3.31) a solution $\Lambda(\Delta F)$ to the so-called compensation equation (3.28) for an unknown functional Λ generating an Sp(2)-doublet λ_a, in order to establish a relation of the Yang–Mills quantum action S_F in a certain gauge determined by a gauge Boson F with the action $S_{F+\Delta F}$ induced by a different gauge $F + \Delta F$. This makes it possible to investigate the problem of gauge-dependence for the generating functional $Z_F(J)$ under a finite change of the gauge in the form (3.35), leading to the gauge-independence of the physical S-matrix.

In terms of the potential Λ inducing the finite field-dependent BRST–antiBRST transformations, we have explicitly constructed (4.24) the parameters λ_a generating a change of the gauge in the path integral for Yang–Mills theories within a class of linear R_ξ-like gauges related to even-valued gauge-fixing functionals $F_{(\xi)}$, with $\xi = 0, 1$ corresponding to the Landau and Feynman (covariant) gauges, respectively. We have shown how to reach an arbitrary gauge given by a gauge Boson F within the path integral representation, starting from the reference frame with a gauge Boson F_0 by means of finite field-dependent BRST–antiBRST transformations with the parameters $\lambda_a(F - F_0)$ given by (4.26).

We have applied the concept of finite field-dependent BRST–antiBRST transformations to construct the Gribov horizon functional h_ξ, given by (5.10) in arbitrary R_ξ-like gauges, starting from a previously known BRST–antiBRST non-invariant functional h, as in [34], corresponding to the Landau gauge and induced by an even-valued functional $F_{(0)}$. The construction is consistent with the study of gauge-independence for the generating functionals of Green's functions $Z_{\mathrm{GZ},F_0}(J)$ in (5.15) within the suggested Gribov–Zwanziger model considered in the BRST–antiBRST approach (5.5).

There are various lines of research for extending the results obtained in the present work. First, the study of finite field-dependent BRST–antiBRST transformations for a general gauge theory in the framework of the path integral[9] (2.12). Second, the development of finite field-dependent BRST transformations for a general gauge theory in the BV quantization method[10] [30]. Third, the construction of finite field-dependent BRST–antiBRST transformations in the Sp(2)-covariant generalized Hamiltonian quantization [12,13] and the study of their properties in connection with the corresponding gauge-fixing problem.[11] Fourth, the consideration of the so-called refined Gribov–Zwanziger theory [47] in a BRST–antiBRST setting analogous to [31], and also the elaboration of a composite operator technique in the BRST–antiBRST Lagrangian quantization scheme, in order to examine the Gribov horizon functional as a composite operator with an external source, along the lines of [39]. We also mention the search for an equivalent local description of the Gribov horizon functional with a set of auxiliary set fields as in [34] such that it should be consistent with both the infinitesimal and finite BRST–antiBRST invariance. We are also interested in the study of the influence of Jacobians generated by finite field-dependent BRST–antiBRST transformations (linear and functionally-independent parameters) on the structure of transformed quantum actions and partition functions [48].

Finally, the suggested Gribov horizon functionals beyond the Landau gauge allow one to study such quantum properties as renormalizability and confinement within the BRST–antiBRST extension of the Gribov–Zwanziger theory in a way consistent with the gauge independence of the physical S-matrix. We intend to study these problems in our forthcoming works.

Concluding, let us outline an ansatz for finite field-dependent BRST–antiBRST transformations of the path integral (2.12), corresponding to the case of a general gauge theory. To this end, notice that the construction (3.5), (3.7) of finite BRST–antiBRST transformations in Section 3, in fact, applies to any infinitesimal symmetry transformations $\delta\phi^A = X^{Aa}\mu_a = (s^a\phi^A)\mu_a$, with anticommuting parameters μ_a, $a = 1, 2$, for a certain functional $S_F(\phi)$, such that $\delta S_F(\phi) = 0$, and does not involve any subsidiary conditions on X^{Aa} and the corresponding s^a, since the construction is achieved only by using $Y^A = (1/2)X^{Aa}_{,B}X^{Bb}\varepsilon_{ba}$ in (3.7), according to (2.20). Let us apply this to the vacuum functional $Z(0)$ of a general gauge theory, given by the path integral (2.12) in the extended space $\Gamma^p = (\phi^A, \phi^*_{Aa}, \bar{\phi}_A, \pi^{Aa}, \lambda^A)$,

$$Z(0) = \int d\Gamma \exp\bigl[(i/\hbar)\mathcal{S}_F(\Gamma)\bigr],$$

$$\mathcal{S}_F = S + \phi^*_{Aa}\pi^{Aa} + (\bar{\phi}_A - F_{,A})\lambda^A - (1/2)\varepsilon_{ab}\pi^{Aa}F_{,AB}\pi^{Bb}, \tag{6.1}$$

9 We have solved this problem in our recent works [43,44].

10 Shortly after the publication of the present work, we have become aware of the more recent study [45] of finite BRST transformations in the BV formalism.

11 We have solved this problem in detail [46], including the case of Yang–Mills theories.

where the integrand $\mathcal{I}_{\Gamma}^{(F)} = d\Gamma \exp[(i/\hbar)\mathcal{S}_F(\Gamma)]$ is invariant, $\delta\mathcal{I}_{\Gamma}^{(F)} = 0$, under the global infinitesimal BRST–antiBRST transformations (2.13), $\delta\Gamma^p = (\sigma^a\Gamma^p)\mu_a$, with the corresponding generators σ^a,

$$\begin{aligned}\delta\Gamma^p &= \left(\sigma^a\Gamma^p\right)\mu_a = \delta\left(\phi^A, \phi^*_{Ab}, \bar{\phi}_A, \pi^{Ab}, \lambda^A\right)\\ &= \left(\pi^{Aa}, \delta^a_b S_{,A}(-1)^{\varepsilon_A}, \varepsilon^{ab}\phi^*_{Ab}(-1)^{\varepsilon_A+1}, \varepsilon^{ab}\lambda^A, 0\right)\mu_a.\end{aligned} \tag{6.2}$$

In this connection, let us determine finite BRST–antiBRST transformations, $\Gamma^p \to \Gamma^p + \Delta\Gamma^p$, parameterized by anticommuting parameters λ_a, $a = 1, 2$, as follows:

$$\begin{aligned}&\mathcal{I}^{(F)}_{\Gamma+\Delta\Gamma} = \mathcal{I}^{(F)}_{\Gamma}, \qquad \left[\frac{\overleftarrow{\partial}}{\partial\lambda_a}\Delta\Gamma^p\right]_{\lambda=0} = \sigma^a\Gamma^p \quad \text{and}\\ &\left[\frac{\overleftarrow{\partial}}{\partial\lambda_a}\frac{\overleftarrow{\partial}}{\partial\lambda_b}\Delta\Gamma^p\right] = \frac{1}{2}\varepsilon^{ab}\sigma^2\Gamma^p, \quad \text{where } \sigma^2 = \sigma_a\sigma^a.\end{aligned} \tag{6.3}$$

Thus determined finite BRST–antiBRST symmetry transformations for the integrand $\mathcal{I}_{\Gamma}^{(F)}$ in a general gauge theory have the form ($\mathcal{X}^{pa} = \sigma^a\Gamma^p$ and $\mathcal{Y}^p = (1/2)\mathcal{X}^{pa}_{,q}\mathcal{X}^{qb}\varepsilon_{ba} = -(1/2)\sigma^2\Gamma^p$)

$$\Delta\Gamma^p = \mathcal{X}^{pa}\lambda_a - \frac{1}{2}\mathcal{Y}^p\lambda^2 = \left(\sigma^a\Gamma^p\right)\lambda_a + \frac{1}{4}\left(\sigma^2\Gamma^p\right)\lambda^2, \qquad \mathcal{I}^{(F)}_{\Gamma+\Delta\Gamma} = \mathcal{I}^{(F)}_{\Gamma}, \tag{6.4}$$

or, in terms of the components,

$$\begin{aligned}&\Delta\phi^A = \pi^{Aa}\lambda_a + \frac{1}{2}\lambda^A\lambda^2, \qquad \Delta\bar{\phi}_A = \varepsilon^{ab}\lambda_a\phi^*_{Ab} + \frac{1}{2}S_{,A}\lambda^2,\\ &\Delta\pi^{Aa} = -\varepsilon^{ab}\lambda^A\lambda_b, \qquad \Delta\lambda^A = 0,\\ &\Delta\phi^*_{Aa} = \lambda_a S_{,A} + \frac{1}{4}(-1)^{\varepsilon_A}\\ &\qquad \times\left[\varepsilon_{ab}\frac{\delta^2 S}{\delta\phi^A\delta\phi^B}\pi^{Bb} + \varepsilon_{ab}\frac{\delta S}{\delta\phi^B}\frac{\delta^2 S}{\delta\phi^A\delta\phi^*_{Bb}}(-1)^{\varepsilon_B} - \phi^*_{Ba}\frac{\delta^2 S}{\delta\phi^A\delta\bar{\phi}_B}(-1)^{\varepsilon_B}\right]\lambda^2.\end{aligned} \tag{6.5}$$

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Appendix A. Group properties of finite BRST–antiBRST transformations

In this appendix, in order to clarify the relations (3.9)–(3.13) of Section 3, we examine the composition of finite variations $\Delta_{(1)}\Delta_{(2)}$ acting on an arbitrary functional $F = F(\phi)$, with the variation ΔF given by (3.8),

$$\Delta F = \left(s^a F\right)\lambda_a + \frac{1}{4}\left(s^2 F\right)\lambda^2. \tag{A.1}$$

Using the readily established Leibnitz-like properties of the generators of BRST–antiBRST transformations, s^a and s^2, acting on the product of any functionals A, B with definite Grassmann parities,

$$s^a(AB) = \big(s^a A\big)B(-1)^{\varepsilon_B} + A\big(s^a B\big) \quad \text{and} \quad s_a(AB) = (s_a A)B(-1)^{\varepsilon_B} + A(s_a B),$$
$$s^2(AB) = \big(s^2 A\big)B - 2(s_a A)\big(s^a B\big)(-1)^{\varepsilon_B} + A\big(s^2 B\big), \quad \text{for } s^2 = s_a s^a, \tag{A.2}$$

and the identities

$$s^a s^b = (1/2)\varepsilon^{ab} s^2 \quad \text{and} \quad s_a s^b = -s^b s_a = (1/2)\delta^b_a s^2 \quad \text{and} \quad s^a s^b s^c \equiv 0, \tag{A.3}$$

with the notation $UV \equiv U_a V^a = -U^a V_a$ for pairing up any Sp(2)-vectors U^a, V^a, we obtain

$$\begin{aligned} s^a(\Delta F) &= s^a\left[\big(s^b F\big)\lambda_b + \frac{1}{4}\big(s^2 F\big)\lambda^2\right] = s^a\big[\big(s^b F\big)\lambda_b\big] + (1/4)s^a\big[\big(s^2 F\big)\lambda^2\big] \\ &= -\big(s^a s^b F\big)\lambda_b + \big(s^b F\big)\big(s^a\lambda_b\big) + (1/4)\big(s^2 F\big)\big(s^a\lambda^2\big) \\ &= -(1/2)\big(s^2 F\big)\lambda^a - (sF)\big(s^a\lambda\big) + (1/4)\big(s^2 F\big)\big(s^a\lambda^2\big) \end{aligned} \tag{A.4}$$

and

$$\begin{aligned} s^2(\Delta F) &= s^2\left[\big(s^b F\big)\lambda_b + \frac{1}{4}\big(s^2 F\big)\lambda^2\right] = s^2\big[\big(s^b F\big)\lambda_b\big] + \frac{1}{4}s^2\big[\big(s^2 F\big)\lambda^2\big] \\ &= 2\big(s_a s^b F\big)\big(s^a\lambda_b\big) + \big(s^b F\big)\big(s^2\lambda_b\big) + \frac{1}{4}\big(s^2 F\big)\big(s^2\lambda^2\big) \\ &= -\big(s^2 F\big)(s\lambda) - (sF)\big(s^2\lambda\big) + \frac{1}{4}\big(s^2 F\big)\big(s^2\lambda^2\big). \end{aligned} \tag{A.5}$$

Therefore, $\Delta_{(1)}\Delta_{(2)}F$ is given by

$$\begin{aligned} \Delta_{(1)}\Delta_{(2)}F &= \big(s^a\Delta_{(2)}F\big)\lambda_{(1)a} + \frac{1}{4}\big(s^2\Delta_{(2)}F\big)\lambda^2_{(1)} \\ &= \big[-(1/2)\big(s^2 F\big)\lambda^a_{(2)} - (sF)\big(s^a\lambda_{(2)}\big) + (1/4)\big(s^2 F\big)\big(s^a\lambda^2_{(2)}\big)\big]\lambda_{(1)a} \\ &\quad + \frac{1}{4}\left[\big(s^2 F\big)(s\lambda_{(2)}) - (sF)\big(s^2\lambda_{(2)}\big) + \frac{1}{4}\big(s^2 F\big)\big(s^2\lambda^2_{(2)}\big)\right]\lambda^2_{(1)} \\ &\equiv \big(s^a F\big)\vartheta_{(1,2)a} + \frac{1}{4}\big(s^2 F\big)\theta_{(1,2)}, \end{aligned} \tag{A.6}$$

whence

$$\vartheta^a_{(1,2)} = -\big(s\lambda^a_{(2)}\big)\lambda_{(1)} + \frac{1}{4}\big(s^2\lambda^a_{(2)}\big)\lambda^2_{(1)}, \tag{A.7}$$

$$\theta_{(1,2)} = \big[2\lambda_{(2)} - \big(s\lambda^2_{(2)}\big)\big]\lambda_{(1)} - \left[(s\lambda_{(2)}) - \frac{1}{4}\big(s^2\lambda^2_{(2)}\big)\right]\lambda^2_{(1)}. \tag{A.8}$$

Hence, the commutator of finite variations reads

$$[\Delta_{(1)}, \Delta_{(2)}]F = \big(s^a F\big)\vartheta_{[1,2]a} + \frac{1}{4}\big(s^2 F\big)\theta_{[1,2]}. \tag{A.9}$$

Finally, using the identity

$$\lambda_{(2)}\lambda_{(1)} - \lambda_{(1)}\lambda_{(2)} = \lambda_{(2)a}\lambda^a_{(1)} - \lambda_{(1)a}\lambda^a_{(2)} = \lambda_{(2)a}\lambda^a_{(1)} - \lambda_{(2)a}\lambda^a_{(1)} \equiv 0, \tag{A.10}$$

we obtain

$$\begin{aligned}
\vartheta^a_{[1,2]} &= \vartheta^a_{(1,2)} - \vartheta^a_{(2,1)} \\
&= \big(s\lambda^a_{(1)}\big)\lambda_{(2)} - \big(s\lambda^a_{(2)}\big)\lambda_{(1)} - \frac{1}{4}\big[\big(s^2\lambda^a_{(1)}\big)\lambda^2_{(2)} - \big(s^2\lambda^a_{(2)}\big)\lambda^2_{(1)}\big],
\end{aligned} \tag{A.11}$$

$$\begin{aligned}
\theta_{[1,2]} = \theta_{(1,2)} - \theta_{(2,1)} = \big[\big(s\lambda^2_{(1)}\big)\lambda_{(2)} - \big(s\lambda^2_{(2)}\big)\lambda_{(1)}\big] + \big[(s\lambda_{(1)})\lambda^2_{(2)} - (s\lambda_{(2)})\lambda^2_{(1)}\big] \\
+ \frac{1}{4}\big[\big(s^2\lambda^2_{(2)}\big)\lambda^2_{(1)} - \big(s^2\lambda^2_{(1)}\big)\lambda^2_{(2)}\big].
\end{aligned} \tag{A.12}$$

In particular, the linear approximation $\Delta^{\mathrm{lin}} F = (s^a F)\lambda_a$, $\Delta F = \Delta^{\mathrm{lin}} F + O(\lambda^2)$, implies (3.13).

Appendix B. Calculation of Jacobians

In this appendix, we present the calculation of the Jacobian (3.14), (3.15), induced in the functional integral (2.21) by the finite BRST–antiBRST transformations (3.7) with an Sp(2)-doublet of anticommuting parameters λ_a, considering the global case, $\lambda_a = \mathrm{const}$, and the case of field-dependent functionals $\lambda_a(\phi)$ of a special form, $\lambda_a(\phi) = s_a \Lambda(\phi)$.

B.1. Constant parameters

Let us assume λ_a to be constant parameters in (3.7) and consider an even matrix M in (3.15) with the elements M^A_B, $\varepsilon(M^A_B) = \varepsilon_A + \varepsilon_B$,

$$\begin{aligned}
M^A_B &= \frac{\delta(\Delta\phi^A)}{\delta\phi^B} = (Q_1)^A_B + R^A_B, \\
&\text{with } (Q_1)^A_B = \frac{\delta X^{Aa}}{\delta\phi^B}\lambda_a(-1)^{\varepsilon_B} \text{ and } R^A_B = -\frac{1}{2}\frac{\delta Y^A}{\delta\phi^B}\lambda^2.
\end{aligned} \tag{B.1}$$

Notice the fact that $Q_1 \sim \lambda_a$, $R \sim \lambda^2$, which, in view of the nilpotency properties $\lambda_a\lambda^2 = \lambda^4 \equiv 0$, implies

$$\mathrm{Str}\big(M^n\big) = \mathrm{Str}(Q_1 + R)^n = \begin{cases} \mathrm{Str}(Q_1 + R) = \mathrm{Str}(R), & n = 1, \\ \mathrm{Str}(Q_1^2) = 2\mathrm{Str}(R), & n = 2, \\ 0, & n > 2. \end{cases} \tag{B.2}$$

Indeed, due to the relations $X^{Aa}_{,A} = 0$ in (2.17), we have

$$\mathrm{Str}(Q_1) = (Q_1)^A_A(-1)^{\varepsilon_A} = \frac{\delta X^{Aa}}{\delta\phi^A}\lambda_a = 0. \tag{B.3}$$

Next, let us examine $\mathrm{Str}(Q_1^2)$:

$$\mathrm{Str}\big(Q_1^2\big) = \big(Q_1^2\big)^A_A(-1)^{\varepsilon_A} = \frac{\delta X^{Aa}}{\delta\phi^B}\lambda_a\frac{\delta X^{Bb}}{\delta\phi^A}\lambda_b(-1)^{\varepsilon_B} = \frac{\delta X^{Aa}}{\delta\phi^B}\frac{\delta X^{Bb}}{\delta\phi^A}\lambda_b\lambda_a(-1)^{\varepsilon_A}. \tag{B.4}$$

Differentiating the relation $X^{Aa}_{,B}X^{Bb} = \varepsilon^{ab}Y^A$ in (2.17) with respect to ϕ^A, we find

$$\frac{\delta}{\delta\phi^B}\left(\frac{\delta X^{Aa}}{\delta\phi^A}\right)X^{Bb}(-1)^{\varepsilon_B} + \frac{\delta X^{Aa}}{\delta\phi^B}\frac{\delta X^{Bb}}{\delta\phi^A} + \varepsilon^{ba}\frac{\delta Y^A}{\delta\phi^A} = 0.$$

Then, due to the relation $X^{Aa}_{,A} = 0$ in (2.17), we have

$$\frac{\delta X^{Aa}}{\delta\phi^B}\frac{\delta X^{Bb}}{\delta\phi^A} = \varepsilon^{ab}\frac{\delta Y^A}{\delta\phi^A}, \tag{B.5}$$

and therefore

$$\mathrm{Str}\left(Q_1^2\right) = \varepsilon^{ab}\frac{\delta Y^A}{\delta\phi^A}\lambda_b\lambda_a(-1)^{\varepsilon_A} = -\frac{\delta Y^A}{\delta\phi^A}\lambda^2(-1)^{\varepsilon_A} = 2\mathrm{Str}(R). \tag{B.6}$$

Thus, the Jacobian $\exp(\Im)$ in (3.15) is given by

$$\Im = -\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\mathrm{Str}\left(M^n\right) = \mathrm{Str}(M) - \frac{1}{2}\mathrm{Str}\left(M^2\right) = \mathrm{Str}(R) - \frac{1}{2}\mathrm{Str}\left(Q_1^2\right) \equiv 0, \tag{B.7}$$

which proves (3.16).

B.2. Field-dependent parameters

In the case of field-dependent parameters $\lambda_a(\phi) = s_a\Lambda(\phi)$ from (3.7), given by an even-valued potential $\Lambda(\phi)$, let us consider an even matrix M in (3.15) with the elements M^A_B,

$$M^A_B \equiv \frac{\delta(\Delta\phi^A)}{\delta\phi^B} = P^A_B + Q^A_B + R^A_B, \quad \text{with } Q^A_B = (Q_1)^A_B + (Q_2)^A_B, \quad \text{for} \tag{B.8}$$

$$P^A_B = X^{Aa}\frac{\delta\lambda_a}{\delta\phi^B}, \qquad (Q_1)^A_B = \lambda_a\frac{\delta X^{Aa}}{\delta\phi^B}(-1)^{\varepsilon_A+1},$$

$$(Q_2)^A_B = \lambda_a Y^A\frac{\delta\lambda^a}{\delta\phi^B}(-1)^{\varepsilon_A+1}, \qquad R^A_B = -\frac{1}{2}\lambda^2\frac{\delta Y^A}{\delta\phi^B}. \tag{B.9}$$

Using the property

$$\mathrm{Str}(AB) = \mathrm{Str}(BA), \tag{B.10}$$

which takes place for any even matrices A, B, and the fact that the occurrence of $R \sim \lambda^2$ in $\mathrm{Str}(M^n)$ more than once yields zero, $\lambda^4 \equiv 0$, we have

$$\mathrm{Str}\left(M^n\right) = \mathrm{Str}(P+Q+R)^n = \sum_{k=0}^{1}C_n^k\mathrm{Str}\left[(P+Q)^{n-k}R^k\right],$$
$$C_n^k = \frac{n!}{k!(n-k)!}. \tag{B.11}$$

Furthermore,

$$\begin{aligned}\mathrm{Str}(P+Q+R)^n &= \mathrm{Str}(P+Q)^n + n\mathrm{Str}\left[(P+Q)^{n-1}R\right]\\ &= \mathrm{Str}(P+Q)^n + n\mathrm{Str}\left(P^{n-1}R\right),\end{aligned} \tag{B.12}$$

since any occurrence of $R \sim \lambda^2$ and $Q \sim \lambda_a$ simultaneously entering $\mathrm{Str}(M)^n$ yields zero, owing to $\lambda_a\lambda^2 = 0$, as a consequence of which R can only be coupled with P^{n-1}.

Having established (B.12), let us examine $\mathrm{Str}(P^{n-1}R)$, namely,

$$\mathrm{Str}\left(P^{n-1}R\right) = \begin{cases}\mathrm{Str}(R), & n = 1,\\ 0, & n > 1.\end{cases} \tag{B.13}$$

Indeed, due to the contraction property $P^2 = f \cdot P \Longrightarrow P^l = f^{l-1} \cdot P$, where f is an even-valued parameter (for details, see (B.34) below), we have

$$\mathrm{Str}\big(P^{n-1}R\big) = f^{n-2}\mathrm{Str}(PR), \quad n > 1, \tag{B.14}$$

$$\begin{aligned}\mathrm{Str}(PR) &= \mathrm{Str}(RP) = (RP)^A_A(-1)^{\varepsilon_A} = R^A_B P^B_A(-1)^{\varepsilon_A}\\ &= -\frac{1}{2}\lambda^2\left(\frac{\delta Y^A}{\delta\phi^B}X^{Bb}\right)\frac{\delta\lambda_b}{\delta\phi^A}(-1)^{\varepsilon_A} = 0,\end{aligned} \tag{B.15}$$

since $Y^A_{,B}X^{Bb} = 0$ in (2.17), which implies

$$\mathrm{Str}\big(M^n\big) = \mathrm{Str}(P+Q)^n + n\mathrm{Str}\big(P^{n-1}R\big) = \begin{cases}\mathrm{Str}(P+Q) + \mathrm{Str}(R), & n = 1,\\ \mathrm{Str}(P+Q)^n, & n > 1,\end{cases} \tag{B.16}$$

so that R drops out of $\mathrm{Str}(M^n)$, $n > 1$, and enters the Jacobian only as $\mathrm{Str}(R)$.

Considering the contribution $\mathrm{Str}(P+Q)^n$ in (B.16), we notice that an occurrence of $Q \sim \lambda_a$ more then twice yields zero, $\lambda_a\lambda_b\lambda_c \equiv 0$. A direct calculation for $n = 2, 3$ leads to

$$\mathrm{Str}(P+Q)^n = \sum_{k=0}^{n} C^k_n \mathrm{Str}\big(P^{n-k}Q^k\big) = \mathrm{Str}\big(P^n + nP^{n-1}Q + C^2_n P^{n-2}Q^2\big). \tag{B.17}$$

Next, starting from the case $n = 4$, $\mathrm{Str}(M^4) = \mathrm{Str}(P^4 + 4P^3Q + 4P^2Q^2 + 2PQPQ)$, one can prove that for any $n \geq 4$ we have

$$\mathrm{Str}(P+Q)^n = \mathrm{Str}\big(P^n + nP^{n-1}Q + nP^{n-2}Q^2 + K_n P^{n-3}QPQ\big), \tag{B.18}$$

where the coefficients[12] K_n are given by (in particular, $n = 4$, $C^2_4 = 6$, $K_4 = C^2_4 - 4 = 2$)

$$K_n = C^2_n - n, \qquad C^2_n = n(n-1)/2 \quad \Longrightarrow \quad K_n = n(n-3)/2, \tag{B.19}$$

which implies

$$\frac{C^2_n}{n} - \frac{K_n}{n} = 1, \qquad \frac{C^2_n}{n} - \frac{K_{n+1}}{n+1} = \frac{1}{2}. \tag{B.20}$$

The proof of (B.18) goes by induction. To this end, suppose that (as in the case $n = 4$)

$$\begin{aligned}&(P+Q)^n = P^n + A^{(1)}_n(P,Q) + B^{(2)}_n(P,Q) + C^{(2)}_n(P,Q), \quad \text{where}\\ &A^{(1)}_n = a_{kl}P^kQP^l, \qquad a_n \equiv a_{k0} = 1, \qquad B^{(2)}_n = b_{kl}P^kQ^2P^l,\\ &C^{(2)}_n = c_{kml}P^kQP^mQP^l, \quad m \geq 1 \quad \text{and} \quad \mathrm{Str}\big(A^{(1)}_n\big) = n\mathrm{Str}\big(P^{n-1}Q\big),\\ &\mathrm{Str}\big(B^{(2)}_n\big) = n\mathrm{Str}\big(P^{n-2}Q^2\big), \qquad \mathrm{Str}\big(C^{(2)}_n\big) = K_n\mathrm{Str}\big(P^{n-3}QPQ\big).\end{aligned} \tag{B.21}$$

Then, due to the vanishing of the terms containing Q more than twice, we have

$$\begin{aligned}&(P+Q)^{n+1} = P^{n+1} + A^{(1)}_{n+1} + B^{(2)}_{n+1} + C^{(2)}_{n+1},\\ &\quad \text{for } A^{(1)}_{n+1} = P^nQ + A^{(1)}_nP,\ B^{(2)}_{n+1} + C^{(2)}_{n+1} = A^{(1)}_nQ + B^{(2)}_nP + C^{(2)}_nP,\end{aligned} \tag{B.22}$$

where

[12] The coefficient K_n turns out to be the number of monomials in $(P+Q)^n$ for $n \geq 4$ that contain two matrices Q and cannot be transformed by cyclic permutations under the symbol Str of supertrace to the form $\mathrm{Str}(P^{n-2}Q^2)$.

$$A^{(1)}_{n+1} = P^n Q + a_{kl} P^k Q P^l P \quad \Longrightarrow \quad a_{n+1} = 1, \tag{B.23}$$

$$B^{(2)}_{n+1} = a_{k0} P^k Q^2 + B^{(2)}_n P, \qquad C^{(2)}_{n+1} = a_{kl} P^k Q P^l Q + C^{(2)}_n P, \quad l \geq 1. \tag{B.24}$$

Due to the contraction property $P^2 = f \cdot P \Longrightarrow P^l = f^{l-1} \cdot P$ in (B.34), the above implies

$$\mathrm{Str}\big(A^{(1)}_{n+1}\big) = (n+1)\mathrm{Str}\big(P^n Q\big), \qquad \mathrm{Str}\big(B^{(2)}_{n+1}\big) = (n+1)\mathrm{Str}\big(P^n Q^2\big), \tag{B.25}$$

$$\mathrm{Str}\big(C^{(2)}_{n+1}\big) = (n-1)\mathrm{Str}\big(P^{n-2} Q P Q\big) + K_n \mathrm{Str}\big(P^{n-2} Q P Q\big). \tag{B.26}$$

Notice that

$$K_n + n - 1 = \frac{n(n-3)}{2} + \frac{2n-2}{2} = \frac{(n+1)(n-2)}{2} = K_{n+1}, \tag{B.27}$$

which proves the induction.

Recall that the Jacobian $\exp(\Im)$ in (3.15) is given by

$$\Im = \mathrm{Str}\ln(\mathbb{I} + M) = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \mathrm{Str}\big(M^n\big), \tag{B.28}$$

where, according to the previous considerations,

$$\mathrm{Str}\big(M^n\big) = \sum_{k=0}^{1} C^k_n \mathrm{Str}\big(P^{n-k} Q^k\big) + D_n, \quad n \geq 1, \tag{B.29}$$

$$\text{for } D_n = \begin{cases} \mathrm{Str}(R), & n = 1, \\ C^2_n \mathrm{Str}(P^{n-2} Q^2), & n = 2, 3, \\ (C^2_n - K_n)\mathrm{Str}(P^{n-2} Q^2) + K_n \mathrm{Str}(P^{n-3} Q P Q), & n > 3, \end{cases} \tag{B.30}$$

or, in detail,

$$\mathrm{Str}\big(M^n\big) = \begin{cases} \mathrm{Str}(P) + \mathrm{Str}(Q) + \mathrm{Str}(R), & n = 1, \\ \mathrm{Str}(P^n) + C^1_n \mathrm{Str}(P^{n-1} Q) + C^2_n \mathrm{Str}(P^{n-2} Q^2), & n = 2, 3, \\ \mathrm{Str}(P^n) + C^1_n \mathrm{Str}(P^{n-1} Q) + (C^2_n - K_n)\mathrm{Str}(P^{n-2} Q^2) & \\ \quad + K_n \mathrm{Str}(P^{n-3} Q P Q), & n > 3. \end{cases} \tag{B.31}$$

First of all, the calculation of the Jacobian is based on the previously established properties (B.6) and (B.3), namely,

$$\mathrm{Str}(Q_1) = 0, \qquad \mathrm{Str}\big(Q_1^2\big) = 2\mathrm{Str}(R). \tag{B.32}$$

It has also been established (Appendix B.1) that the quantity $\mathrm{Str}(R)$ in (B.16) cancels the contribution $\mathrm{Str}(Q_1^2)$ to the Jacobian, where these contributions enter in the first and second orders, $\mathrm{Str}(M^1)$ and $\mathrm{Str}(M^2)$, respectively, thus summarily producing an identical zero:

$$\mathrm{Str}(R) - (1/2)\mathrm{Str}\big(Q_1^2\big) \equiv 0. \tag{B.33}$$

Therefore, we can exclude $\mathrm{Str}(R)$ and $\mathrm{Str}(Q_1^2)$ from further consideration.

Recalling that $\lambda_a = s_a \Lambda$, we can deduce the additional properties

$$P^2 = f \cdot P, \qquad Q P = (1 + f) \cdot Q_2, \qquad f = -\frac{1}{2}\mathrm{Str}(P), \tag{B.34}$$

where the quantity f is given by

$$\frac{\delta\lambda_b}{\delta\phi^A}X^{Aa} = s^a\lambda_b = \delta^a_b f \quad \Longrightarrow \quad f = \frac{1}{2}s^a\lambda_a = -\frac{1}{2}s^2\Lambda. \tag{B.35}$$

Indeed,

$$\begin{aligned}
&\left(P^2\right)^A_B = (P)^A_D(P)^D_B = X^{Aa}\left(\frac{\delta\lambda_a}{\delta\phi^D}X^{Db}\right)\frac{\delta\lambda_b}{\delta\phi^B} = f\cdot\delta^b_a X^{Aa}\frac{\delta\lambda_b}{\delta\phi^B} = f\cdot(P)^A_B,\\
&\frac{\delta\lambda_a}{\delta\phi^B}X^{Bb} = s^b\lambda_a = s^b s_a\Lambda = \delta^b_a f, \qquad f = \Lambda_{,A}Y^A - (1/2)\varepsilon_{ab}X^{Aa}\Lambda_{,AB}X^{Bb},\\
&f = \frac{1}{2}\left(\frac{\delta\lambda_a}{\delta\phi^A}X^{Aa}\right) = -\frac{1}{2}(P)^A_A(-1)^{\varepsilon_A} = -\frac{1}{2}\mathrm{Str}(P).
\end{aligned} \tag{B.36}$$

As a consequence, we have $QP = (1+f)\cdot Q_2$, namely, in view of $X^{Aa}_{,B}X^{Bb} = \varepsilon^{ab}Y^A$ from (2.17),

$$\begin{aligned}
(QP)^A_B = Q^A_D P^D_B &= (-1)^{\varepsilon_A+1}\lambda_a\left(\frac{\delta X^{Aa}}{\delta\phi^D} + Y^A\frac{\delta\lambda^a}{\delta\phi^D}\right)X^{Dd}\frac{\delta\lambda_d}{\delta\phi^B}\\
&= (-1)^{\varepsilon_A+1}\lambda_a\left[\varepsilon^{ab}Y^A + Y^A\left(s^b\lambda^a\right)\right]\frac{\delta\lambda_b}{\delta\phi^B}\\
&= (-1)^{\varepsilon_A+1}\lambda_a\left[\varepsilon^{ab}Y^A + \varepsilon^{ad}Y^A\delta^b_d f\right]\frac{\delta\lambda_b}{\delta\phi^B}\\
&= (-1)^{\varepsilon_A+1}\lambda_a Y^A(1+f)\frac{\delta\lambda^a}{\delta\phi^B} = (1+f)(Q_2)^A_B.
\end{aligned} \tag{B.37}$$

Finally,

$$\begin{aligned}
&\mathrm{Str}\left(P^n\right) = f^{n-1}\mathrm{Str}(P) = -2f^n, \quad n\geq 1,\\
&\mathrm{Str}\left(P^{n-1}Q\right) = \begin{cases}\mathrm{Str}(Q) = \mathrm{Str}(Q_2), & n=1,\\ f^{n-2}\mathrm{Str}(PQ) = f^{n-2}\mathrm{Str}(QP) = f^{n-2}(1+f)\mathrm{Str}(Q_2), & n>1,\end{cases}\\
&\mathrm{Str}\left(P^{n-2}Q^2\right) = \begin{cases}\mathrm{Str}(Q^2) = \mathrm{Str}(2Q_1Q_2 + Q_2^2), & n=2,\\ f^{n-3}\mathrm{Str}(PQ^2) = f^{n-3}\mathrm{Str}[Q(QP)] & \\ \qquad = f^{n-3}(1+f)\mathrm{Str}[(Q_1+Q_2)Q_2], & n>2,\end{cases}\\
&\mathrm{Str}\left(P^{n-3}QPQ\right) = f^{n-4}\mathrm{Str}(PQPQ) = f^{n-4}\mathrm{Str}\left[(QP)(QP)\right]\\
&\qquad = f^{n-4}(1+f)^2\mathrm{Str}\left(Q_2^2\right), \quad n>3,
\end{aligned} \tag{B.38}$$

where the term $\mathrm{Str}(Q_1^2)$ has been omitted according to the previous considerations related to (B.33).

We further notice that $\mathrm{Str}(Q_1Q_2)\not\equiv 0$. Indeed, due to $X^{Aa}_{,B}X^{Bb} = \varepsilon^{ab}Y^A$ and $Y^A_{,B}X^{Bb} = 0$ in (2.17), we have

$$\begin{aligned}
&(Q_1Q_2)^A_A(-1)^{\varepsilon_A}\\
&\quad = \lambda_a\frac{\delta X^{Aa}}{\delta\phi^B}Y^B\frac{\delta\lambda^2}{\delta\phi^A} = \frac{1}{2}\lambda_a\left(\frac{\delta X^{Aa}}{\delta\phi^B}\frac{\delta X^{Bb}}{\delta\phi^D}\right)X^{Dd}\varepsilon_{db}\frac{\delta\lambda^2}{\delta\phi^A}\\
&\quad = \frac{1}{2}\lambda_a\left[\frac{\delta}{\delta\phi^D}\left(\frac{\delta X^{Aa}}{\delta\phi^B}X^{Bb}\right) - \left(\frac{\delta}{\delta\phi^D}\frac{\delta X^{Aa}}{\delta\phi^B}\right)X^{Bb}(-1)^{\varepsilon_D(\varepsilon_B+1)}\right]X^{Dd}\varepsilon_{db}\frac{\delta\lambda^2}{\delta\phi^A}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\lambda_a\left[\varepsilon^{ab}\frac{\delta Y^A}{\delta\phi^D}X^{Dd} - \left(\frac{\delta}{\delta\phi^D}\frac{\delta X^{Aa}}{\delta\phi^B}\right)X^{Bb}X^{Dd}(-1)^{\varepsilon_D(\varepsilon_B+1)}\right]\varepsilon_{db}\frac{\delta\lambda^2}{\delta\phi^A}\\
&= \frac{1}{2}\left(X^{Bb}\frac{\delta^2 X^{Aa}}{\delta\phi^D\delta\phi^B}X^{Dd}\varepsilon_{db}\right)\lambda_a\frac{\delta\lambda^2}{\delta\phi^A}. \qquad \text{(B.39)}
\end{aligned}
$$

Besides,

$$
\operatorname{Str}\left(Q_2^2\right) = \operatorname{Str}^2(Q_2) \not\equiv 0. \qquad \text{(B.40)}
$$

Indeed,

$$
(Q_2)^A_A(-1)^{\varepsilon_A} = \lambda_a Y^A\frac{\delta\lambda^a}{\delta\phi^A}, \qquad \text{(B.41)}
$$

$$
(Q_2)^A_B(Q_2)^B_A(-1)^{\varepsilon_A} = \left(\lambda_a Y^B\frac{\delta\lambda^a}{\delta\phi^B}\right)\left(\lambda_b Y^A\frac{\delta\lambda^b}{\delta\phi^A}\right). \qquad \text{(B.42)}
$$

Therefore, $\Im$ in the expression (B.28) for the Jacobian $\exp(\Im)$ has the general structure

$$
\begin{aligned}
&\Im = A(f) + B(f|Q_2) + C(f|Q_1Q_2),\\
&\quad \text{for } B(f|Q_2) = b_1(f)\operatorname{Str}(Q_2) + b_2(f)\operatorname{Str}\left(Q_2^2\right) = \left[b_1(f) + b_2(f)\operatorname{Str}(Q_2)\right]\operatorname{Str}(Q_2),\\
&\quad \text{and } C(f|Q_1Q_2) = c(f)\operatorname{Str}(Q_1Q_2). \qquad \text{(B.43)}
\end{aligned}
$$

Let us examine $A(f)$, namely,

$$
A(f) = -\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\operatorname{Str}\left(P^n\right) = 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}f^n = -2\ln(1+f). \qquad \text{(B.44)}
$$

Let us examine the explicit structure of the series related to $b_1(f)$: the quantity $\operatorname{Str}(Q_2)$ derives from $\operatorname{Str}(P^{n-1}Q)$ for $n \geq 1$ in (B.38), and is coupled with the combinatorial coefficient C^1_n. The part of $\Im$ containing $\operatorname{Str}(Q_2)$ is given by

$$
b_1(f)\operatorname{Str}(Q_2) = C^1_1\operatorname{Str}(Q_2) - \sum_{n=2}^{\infty}\frac{(-1)^n}{n}C^1_n f^{n-2}(1+f)\operatorname{Str}(Q_2), \qquad \text{(B.45)}
$$

whence

$$
b_1(f) = 1 - (1+f)\sum_{m=0}^{\infty}(-1)^m f^m = 1 - (1+f)(1+f)^{-1} \equiv 0. \qquad \text{(B.46)}
$$

Let us examine the explicit structure of the series related to $b_2(f)$: the quantity $\operatorname{Str}^2(Q_2)$ derives from $\operatorname{Str}(P^{n-2}Q^2)$ for $n \geq 2$ in (B.38), coupled with the combinatorial coefficients C^2_n for $n = 2, 3$ and $(C^2_n - K_n)$ for $n > 3$, and also derives from $\operatorname{Str}(P^{n-3}QPQ)$ for $n > 3$ in (B.38), coupled with the combinatorial coefficients K_n. The part of $\Im$ containing $\operatorname{Str}^2(Q_2)$ reads

$$
\begin{aligned}
b_2(f)\operatorname{Str}^2(Q_2) = {} & -\frac{(-1)^2}{2}C^2_2\operatorname{Str}^2(Q_2) - \frac{(-1)^3}{3}C^2_3(1+f)\operatorname{Str}^2(Q_2)\\
& - \sum_{n=4}^{\infty}\frac{(-1)^n}{n}\left(C^2_n - K_n\right)f^{n-3}(1+f)\operatorname{Str}^2(Q_2)\\
& - \sum_{n=4}^{\infty}\frac{(-1)^n}{n}K_n f^{n-4}(1+f)^2\operatorname{Str}^2(Q_2), \qquad \text{(B.47)}
\end{aligned}
$$

whence

$$b_2(f) = -\frac{1}{2} + (1+f) - \sum_{n=4}^{\infty} \frac{(-1)^n}{n} \big[\big(C_n^2 - K_n\big) f^{n-3}(1+f) + K_n f^{n-4}(1+f)^2 \big]$$
$$= \frac{1}{2} + f - (1+f) \sum_{n=4}^{\infty} \frac{(-1)^n}{n} \big(C_n^2 f^{n-3} + K_n f^{n-4}\big)$$
$$= \frac{1}{2} + f - (1+f) \left[\frac{1}{2} - \sum_{m=1}^{\infty} (-1)^m \left(\frac{C_{m+3}^2}{m+3} - \frac{K_{m+4}}{m+4} \right) f^m \right]. \tag{B.48}$$

By virtue of (B.20), this implies the vanishing of $b_2(f)$, namely,

$$b_2(f) = \frac{1}{2} f + (1+f) \sum_{m=1}^{\infty} (-1)^m \left(\frac{1}{2}\right) f^m = \frac{1}{2} f + \frac{1}{2}(1+f) \sum_{m=1}^{\infty} (-1)^m f^m$$
$$= \frac{1}{2} f + \frac{1}{2}(1+f)\big[(1+f)^{-1} - 1\big] \equiv 0. \tag{B.49}$$

Let us examine the explicit structure of the series related to $c(f)$: the quantity $\mathrm{Str}(Q_1 Q_2)$ derives from $\mathrm{Str}(P^{n-2} Q^2)$ for $n \geq 2$ in (B.38), and is coupled with the combinatorial coefficients C_n^2, for $n = 2, 3$, and $C_n^2 - K_n$, for $n > 3$. The part of $\Im$ containing $\mathrm{Str}(Q_1 Q_2)$ is given by

$$c(f)\mathrm{Str}(Q_1 Q_2) = -\frac{(-1)^2}{2} C_2^2 \mathrm{Str}(2 Q_1 Q_2) - \frac{(-1)^3}{3} C_3^2 (1+f) \mathrm{Str}(Q_1 Q_2)$$
$$- \sum_{n=4}^{\infty} \frac{(-1)^n}{n} \big(C_n^2 - K_n\big) f^{n-3}(1+f) \mathrm{Str}(Q_1 Q_2), \tag{B.50}$$

whence

$$c(f) = -1 + (1+f) - \sum_{n=4}^{\infty} \frac{(-1)^n}{n} \big(C_n^2 - K_n\big) f^{n-3}(1+f)$$
$$= f - (1+f) \sum_{n=4}^{\infty} (-1)^n \left(\frac{C_n^2}{n} - \frac{K_n}{n} \right) f^{n-3}. \tag{B.51}$$

By virtue of (B.20), this implies the vanishing of $c(f)$, namely,

$$c(f) = f - (1+f) \sum_{n=4}^{\infty} (-1)^n f^{n-3} = f + (1+f) \sum_{m=1}^{\infty} (-1)^m f^m$$
$$= f + (1+f)\big[(1+f)^{-1} - 1\big] \equiv 0. \tag{B.52}$$

From the vanishing of all the coefficients $b_1(f)$, $b_2(f)$, $c(f)$, due to (B.46), (B.49), (B.52), we conclude that

$$B(f|Q_2) = b_1(f)\mathrm{Str}(Q_2) + b_2(f)\mathrm{Str}\big(Q_2^2\big) \equiv 0 \quad \text{and}$$
$$C(f|Q_1 Q_2) = c(f)\mathrm{Str}(Q_1 Q_2) \equiv 0, \tag{B.53}$$

and therefore the Jacobian $\exp(\Im)$ is finally given by

$$\begin{aligned}\Im &= A(f) + B(f|Q_2) + C(f|Q_1Q_2) = A(f)\\ &= -2\ln(1+f) \quad \text{for } f = -(1/2)s^2\Lambda,\end{aligned} \tag{B.54}$$

which is identical with (3.17).

Appendix C. BRST–antiBRST invariant Yang–Mills action in R_ξ-like gauges

In this appendix, we present the details of calculations used in Section 4 to establish a correspondence between the gauge-fixing procedures in the Yang–Mills theory described by a gauge-fixing function $\chi(\phi) = 0$ from the class of R_ξ-gauges in the BV formalism [30] and by a gauge-fixing functional F in the BRST–antiBRST quantization [15,16].

The Yang–Mills theories belong to the class of irreducible gauge theories of rank 1 with a closed algebra, which implies that $M^{ij}_{\alpha\beta} = 0$ in (2.3) and that any solution of the equation $R^i_\alpha X^\alpha = 0$ has the form $X^\alpha = 0$. The corresponding space of fields and antifields $(\phi^A, \phi^*_{Aa}, \bar{\phi})$ is given by

$$\phi^A = \left(A^i, B^\alpha, C^{\alpha a}\right), \qquad \phi^*_{Aa} = \left(A^*_{ia}, B^*_{\alpha a}, C^*_{\alpha ab}\right), \qquad \bar{\phi} = (\bar{A}_i, \bar{B}_\alpha, \bar{C}_{\alpha a}), \tag{C.1}$$

as we take into account (2.1) and the following distribution of the Grassmann parity and ghost number:

$$\varepsilon\left(\phi^A\right) \equiv (\varepsilon_i, \varepsilon_\alpha, \varepsilon_\alpha + 1), \qquad \mathrm{gh}\left(\phi^A\right) = \left(0, 0, (-1)^{a+1}\right), \tag{C.2}$$

whereas a solution to the generating equations (2.5) with a vanishing right-hand side can be found in the linear form (2.16), $S = S_0 + \phi^*_{Aa} X^{Aa} + \bar{\phi}_A Y^A$, obviously satisfying the boundary condition $S|_{\phi^* = \bar{\phi} = 0} = S_0$. Here, the functionals X^{Aa} and Y^A can be chosen as [15]

$$X^{Aa} = \left(X^{ia}_1, X^{\alpha a}_2, X^{\alpha ab}_3\right), \qquad Y^A = \left(Y^i_1, Y^\alpha_2, Y^{\alpha a}_3\right), \tag{C.3}$$

where

$$\begin{aligned}
X^{ia}_1 &= R^i_\alpha C^{\alpha a},\\
X^{\alpha a}_2 &= -\frac{1}{2} F^\alpha_{\gamma\beta} B^\beta C^{\gamma a} - \frac{1}{12}(-1)^{\varepsilon_\beta}\left(2F^\alpha_{\gamma\beta,j} R^j_\rho + F^\alpha_{\gamma\sigma} F^\sigma_{\beta\rho}\right) C^{\rho b} C^{\beta a} C^{\gamma c} \varepsilon_{cb},\\
X^{\alpha ab}_3 &= -\varepsilon^{ab} B^\alpha - \frac{1}{2}(-1)^{\varepsilon_\beta} F^\alpha_{\beta\gamma} C^{\gamma b} C^{\beta a},\\
Y^i_1 &= R^i_\alpha B^\alpha + \frac{1}{2}(-1)^{\varepsilon_\alpha} R^i_{\alpha,j} R^j_\beta C^{\beta b} C^{\alpha a} \varepsilon_{ab},\\
Y^\alpha_2 &= 0, \qquad Y^{\alpha a}_3 = -2X^{\alpha a}_3.
\end{aligned} \tag{C.4}$$

By construction, the functionals $X^{Aa} = \delta S / \delta \phi^*_{Aa}$ and $Y^A = \delta S / \delta \bar{\phi}_A$ obey the properties $S_{0,i} X^{ia} = 0$, $X^{Aa}_{,B} X^{Bb} = \varepsilon^{ab} Y^A$, $Y^B_{,A} X^{Aa} = 0$. Besides, in Yang–Mills theories the explicit form (4.2), (4.3) of the gauge generators R^i_α and structure coefficients $F^\gamma_{\alpha\beta} = \text{const}$ is such that $X^{Aa} = (X^{ia}_1, X^{\alpha a}_2, X^{\alpha ab}_3)$ in (C.4) possess the properties $X^{Aa}_{,A} = 0$, so that the entire set of relations (2.17) is fulfilled, and the solution given by (C.4) satisfies the generating equations (2.5) identically.

As we keep the following consideration restricted to the case of constant structure coefficients, $F^\alpha_{\beta\gamma,j} = 0$, let us choose the gauge-fixing functional $F(\phi)$ in the form

$$F = F(A, C), \qquad \frac{\delta^2 F}{\delta A^i \delta A^j} \neq 0, \qquad \frac{\delta^2 F}{\delta C^{\alpha a} \delta C^{\alpha a}} \neq 0. \tag{C.5}$$

By virtue of (C.4), the quantum action $S_F(\phi)$ in (2.22) reads as follows:

$$\begin{aligned} S_F = S_0 &+ \frac{\delta F}{\delta A^i}\left(R^i_\alpha B^\alpha + \frac{1}{2}(-1)^{\varepsilon_\alpha} R^i_{\alpha,j} R^j_\beta C^{\beta b} C^{\alpha a}\varepsilon_{ab}\right) \\ &- \frac{1}{2}\varepsilon_{ab}\left(R^i_\alpha C^{\alpha a}\right)\frac{\delta^2 F}{\delta A^i \delta A^j}\left(R^j_\beta C^{\beta b}\right) \\ &+ \frac{\delta F}{\delta C^{\alpha a}}\left(F^\alpha_{\gamma\beta} B^\beta C^{\gamma a} + \frac{1}{6}(-1)^{\varepsilon_\beta} F^\alpha_{\gamma\sigma} F^\sigma_{\beta\rho} C^{\rho b} C^{\beta a} C^{\gamma c}\varepsilon_{cb}\right) \\ &- \frac{1}{2}\varepsilon_{ab}\left(\varepsilon^{ac} B^\alpha + \frac{1}{2}(-1)^{\varepsilon_\gamma} F^\alpha_{\gamma\delta} C^{\delta c} C^{\gamma a}\right) \\ &\times \frac{\delta^2 F}{\delta C^{\alpha c}\delta C^{\beta d}}\left(\varepsilon^{bd} B^\beta + \frac{1}{2}(-1)^{\varepsilon_\rho} F^\beta_{\rho\sigma} C^{\sigma d} C^{\rho b}\right). \end{aligned} \tag{C.6}$$

Using the identity

$$\begin{aligned} &\frac{\delta F}{\delta A^i} R^i_\alpha B^\alpha + \frac{1}{2}(-1)^{\varepsilon_\alpha}\varepsilon_{ab}\frac{\delta F}{\delta A^i} R^i_{\alpha,j} R^j_\beta C^{\beta b} C^{\alpha a} - \frac{1}{2}\varepsilon_{ab}\left(R^i_\alpha C^{\alpha a}\right)\frac{\delta^2 F}{\delta A^i \delta A^j}\left(R^j_\beta C^b\right) \\ &\quad = \chi_\alpha B^\alpha + \frac{1}{2}(-1)^{\varepsilon_\alpha}\left(\chi_{\alpha,i} R^i_\beta\right) C^{\beta b} C^{\alpha a}\varepsilon_{ab}, \quad \text{for } \chi_\alpha \equiv \frac{\delta F}{\delta A^i} R^i_\alpha, \end{aligned} \tag{C.7}$$

we obtain

$$\begin{aligned} S_F = S_0 &+ \frac{\delta F}{\delta A^i}\mathcal{A}^i - \frac{1}{2}\varepsilon_{ab}\left[\frac{\delta}{\delta A^j}\left(\frac{\delta F}{\delta A^i}\mathcal{A}^{ia}\right)\right]\mathcal{A}^{jb} + \frac{\delta F}{\delta C^{\alpha a}}\mathcal{C}^{\alpha a} \\ &- \frac{1}{2}\varepsilon_{ab}\mathcal{C}^{\alpha ac}\left(\frac{\delta}{\delta C^{\beta d}}\frac{\delta F}{\delta C^{\alpha c}}\right)\mathcal{C}^{\beta bd}, \end{aligned} \tag{C.8}$$

where

$$\begin{aligned} &\mathcal{A}^i \equiv R^i_\alpha B^\alpha, \qquad \mathcal{A}^{ia} \equiv R^i_\alpha C^{\alpha a}, \\ &\mathcal{C}^{\alpha a} \equiv F^\alpha_{\gamma\beta} B^\beta C^{\gamma a} + \frac{1}{6}(-1)^{\varepsilon_\beta} F^\alpha_{\gamma\sigma}\left(F^\sigma_{\beta\rho} C^{\rho b} C^{\beta a}\right) C^{\gamma c}\varepsilon_{cb}, \\ &\mathcal{C}^{\alpha ab} \equiv \varepsilon^{ab} B^\alpha + \frac{1}{2}(-1)^{\varepsilon_\beta} F^\alpha_{\beta\gamma} C^{\gamma b} C^{\beta a}, \quad \text{with } \varepsilon\left(\mathcal{A}^i\right) = \varepsilon\left(\mathcal{A}^{ia}\right) + 1 = \varepsilon_i, \\ &\varepsilon\left(\mathcal{C}^{\alpha ab}\right) = \varepsilon\left(\mathcal{C}^{\alpha a}\right) + 1 = \varepsilon_\alpha. \end{aligned} \tag{C.9}$$

For Yang–Mills theories, with the classical action S_0, gauge generators R^i_α and structure coefficients $F^\gamma_{\alpha\beta}$ given by (4.1), (4.2), (4.3), and with the set of fields ϕ^A given by (4.4), (4.5), the relations (C.8), (C.9) take the form

$$\begin{aligned} S_F = S_0 &+ \int d^D x \left\{\frac{\delta F}{\delta A^{m\mu}}\mathcal{A}^{m\mu} - \frac{1}{2}\varepsilon_{ab}\left[\frac{\delta}{\delta A^{n\nu}}\left(\frac{\delta F}{\delta A^{m\mu}}\mathcal{A}^{m\mu a}\right)\mathcal{A}^{n\nu b}\right]\right\} \\ &+ \int d^D x \left[\frac{\delta F}{\delta C^{ma}}\mathcal{C}^{ma} - \frac{1}{2}\varepsilon_{ab}\mathcal{C}^{mac}\left(\frac{\delta}{\delta C^{nd}}\frac{\delta F}{\delta C^{mc}}\right)\mathcal{C}^{nbd}\right], \end{aligned} \tag{C.10}$$

where

$$\mathcal{A}^m_\mu \equiv D^{mn}_\mu B^n, \qquad \mathcal{A}^{ma}_\mu \equiv D^{mn}_\mu C^{na},$$
$$\mathcal{C}^{ma} \equiv f^{mnl} B^l C^{na} + \frac{1}{6} f^{mnl}\big(f^{lrs} C^{sb} C^{ra}\big) C^{nc} \varepsilon_{cb},$$
$$\mathcal{C}^{mab} \equiv \varepsilon^{ab} B^m + \frac{1}{2} f^{mnl} C^{lb} C^{na}, \qquad \varepsilon\big(\mathcal{A}^m_\mu\big) = \varepsilon\big(\mathcal{A}^{ma}_\mu\big) + 1 = 0,$$
$$\varepsilon\big(\mathcal{C}^{ma}\big) = \varepsilon\big(\mathcal{C}^{mab}\big) + 1 = 1. \tag{C.11}$$

Choosing the gauge-fixing functional $F(A, C)$ in the quadratic form (4.11) and using the identities (for arbitrary $su(N)$-vectors F^m and G^m)

$$D^{mn}_\mu A^{n\mu} = \partial_\mu A^{m\mu}, \qquad \int d^D x \big(D^{mn}_\mu F^n\big) G^m = -\int d^D x\, F^m D^{mn}_\mu G^n, \tag{C.12}$$

we have

$$\delta_A F = -\alpha \int d^D x\, A^m_\mu \delta A^{m\mu}, \tag{C.13}$$

$$\frac{\delta F}{\delta A^{m\mu}} \mathcal{A}^{m\mu} = -\alpha \int d^D x\, A^{m\mu} D^{mn}_\mu B^n = \alpha \int d^D x\, \big(D^{nm}_\mu A^{m\mu}\big) B^n$$
$$= \alpha \int d^D x \big(\partial_\mu A^{m\mu}\big) B^{mn}, \tag{C.14}$$

$$\frac{\delta F}{\delta A^{m\mu}} \mathcal{A}^{m\mu a} = -\alpha \int d^D x\, A^{m\mu} D^{mn}_\mu C^{na} = \alpha \int d^D x \big(\partial_\mu A^{n\mu}\big) C^{na}, \tag{C.15}$$

whence

$$\delta_A \left(\frac{\delta F}{\delta A^{m\mu}} \mathcal{A}^{m\mu a}\right) = \alpha \int d^D x \big(\partial_\mu \delta A^{m\mu}\big) C^{ma} = -\alpha \int d^D x \big(\partial_\mu C^{ma}\big) \delta A^{m\mu},$$
$$\int d^D x \left[\frac{\delta}{\delta A^{n\nu}} \left(\frac{\delta F}{\delta A^{m\mu}} \mathcal{A}^{m\mu a}\right)\right] \mathcal{A}^{n\nu b} = -\alpha \int d^D x \big(\partial_\mu C^{ma}\big) D^{mn\mu} C^{nb}. \tag{C.16}$$

Next,

$$\delta_C F = -\beta \varepsilon_{ba} \int d^D x\, C^{mb} \delta C^{ma} \quad \Longrightarrow \quad \frac{\delta F}{\delta C^{ma}} = \beta \varepsilon_{ab} C^{mb}, \tag{C.17}$$

$$\int d^D x \frac{\delta F}{\delta C^{ma}} \mathcal{C}^{ma} = \beta \varepsilon_{ab} \int d^D x\, C^{mb} \mathcal{C}^{ma}$$
$$= \beta \varepsilon_{ba} \int d^D x\, C^{ma} \left(f^{mnl} B^l C^{nb} + \frac{1}{6} f^{mnl} f^{lrs} C^{sd} C^{rb} C^{nc} \varepsilon_{cd}\right). \tag{C.18}$$

At the same time,

$$\delta_C \left(\frac{\delta F}{\delta C^{mc}(x)}\right) = \beta \varepsilon_{cd} \delta C^{md}(x) = \beta \varepsilon_{cd} \int d^D y\, \delta^{mn} \delta(y - x) \delta C^{nd}(y),$$
$$\frac{\delta}{\delta C^{nd}(y)} \left(\frac{\delta F}{\delta C^{mc}(x)}\right) = \beta \varepsilon_{cd} \delta^{mn} \delta(y - x), \tag{C.19}$$

whence

$$
\begin{aligned}
&-\frac{1}{2}\varepsilon_{ab}\int d^D x\, d^D y\, C^{mac}(x)\frac{\delta}{\delta C^{nd}(y)}\left(\frac{\delta F}{\delta C^{mc}(x)}\right)C^{nbd}(y) \\
&\quad = -\frac{1}{2}\varepsilon_{ab}\int d^D x\, d^D y\, C^{mac}(x)\left[\beta\varepsilon_{cd}\delta^{mn}\delta(y-x)\right]C^{nbd}(y) \\
&\quad = -\frac{\beta}{2}\varepsilon_{ab}\varepsilon_{cd}\int d^D x\left(\varepsilon^{ac}B^m + \frac{1}{2}f^{mnl}C^{lc}C^{na}\right)\left(\varepsilon^{bd}B^m + \frac{1}{2}f^{mrs}C^{sd}C^{rb}\right). \qquad \text{(C.20)}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\int d^D x\left[\frac{\delta F}{\delta C^{ma}}C^{ma} - \frac{1}{2}\varepsilon_{ab}C^{mac}\frac{\delta}{\delta C^{nd}}\left(\frac{\delta F}{\delta C^{mc}}\right)C^{nbd}\right] \\
&\quad = -\beta\varepsilon_{ab}\int d^D x\, C^{ma}\left(f^{mnl}B^l C^{mb} + \frac{1}{6}f^{mnl}f^{lrs}C^{sd}C^{rb}C^{nc}\varepsilon_{cd}\right) \\
&\qquad - \frac{\beta}{2}\varepsilon_{ab}\varepsilon_{cd}\int d^D x\left(\varepsilon^{ac}B^m + \frac{1}{2}f^{mnl}C^{lc}C^{na}\right)\left(\varepsilon^{bd}B^m + \frac{1}{2}f^{mrs}C^{sd}C^{rb}\right). \qquad \text{(C.21)}
\end{aligned}
$$

Finally,

$$
S_F(A,B,C) = S_0(A) + S_1(A,B) + S_2(A,C) + S_3(A,B,C), \qquad \text{(C.22)}
$$

where

$$
\begin{aligned}
&S_1 = \alpha\int d^D x\left(\partial^\mu A^m_\mu\right)B^m, \qquad S_2 = \frac{\alpha}{2}\varepsilon_{ab}\int d^D x\left(\partial^\mu C^{ma}\right)D^{mn}_\mu C^{nb}, \\
&S_3 = -\beta\varepsilon_{ab}\int d^D x\, C^{ma}\left(f^{mnl}B^l C^{mb} + \frac{1}{6}f^{mnl}f^{lrs}C^{sd}C^{rb}C^{nc}\varepsilon_{cd}\right) \\
&\qquad - \frac{\beta}{2}\varepsilon_{ab}\varepsilon_{cd}\int d^D x\left(\varepsilon^{ac}B^m + \frac{1}{2}f^{mnl}C^{lc}C^{na}\right)\left(\varepsilon^{bd}B^m + \frac{1}{2}f^{mrs}C^{sd}C^{rb}\right). \qquad \text{(C.23)}
\end{aligned}
$$

By virtue of the identity $f^{lmn}C^{nb}C^{ma}\varepsilon_{ab} \equiv 0$, the quantum action (C.22) equals to (4.12).

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